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THE ELEMENTS

OF

OPTICS:

DESIGNED FOR THE USE OF STUDENTS
IN THE UNIVERSITY.

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CONTENTS.

THE ELEMENTS

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OPTICS.

Sect. I.

ON THE NATURE OF LIGHT, AND THE LAWS OF REFLECTION AND REFRACTION.

(Art. 1.) By Optics we understand that branch of Natural Philosophy which treats of the nature and properties of Light, and the Theory of Vision.

(2.) Modern Philosophers have made two hypotheses to explain the manner in which vision is produced by luminous objects. Des Cartes, Huygens and Euler, suppose that there is a subtile, elastic medium which penetrates all bodies, and fills all space; and that vibrations, excited in this fluid by the luminous body, are propagated thence to the eye, and produce the sensation of vision, in the same manner that the vibrations of the air, striking against the ear, produce the sensation of sound.

It has been objected to this hypothesis, and the objection has never been answered, that the vibrations of an elastic fluid are propagated in every direction, and into every corner to which the fluid extends; on the supposition therefore that light is nothing more than

the effect of the vibrations of such a fluid, there could be no shadow, or darkness.

If it be said that the fluid, by means of which vision is excited, is different from all other elastic fluids, the effect is ascribed to a cause, the nature of which is unknown; and the hypothesis amounts to nothing more than a confession, that we are ignorant in what manner vision is produced.

The other hypothesis, adopted by Sir I. Newton and his followers, is, that light consists of very small particles of matter, which are constantly thrown off from luminous bodies, and which produce the sensation of vision by actual impact upon the proper organ.

In favour of this hypothesis, it is observed that the motion of light is conformable to the laws which regulate the motions of small bodies, under the same circumstances: Thus, where it meets with no impediment, it moves uniformly forward in right lines*; and in it's passage into, and reflection from different mediums, the direction of it's motion is changed as it would be, did it consist of small particles of matter, attracted towards, or repelled from the surfaces upon which they are incident†.

Whether light has other properties of matter or not, is a question which does not appear to have been fairly decided; we may however be allowed to consider it as material, and to speak of it as consisting of particles of *matter*, till a more satisfactory hypothesis can be framed; especially, as we deduce no conclusions from the supposition, nor build any theory upon it. Those properties of light from which our theory of vision is

* See *Mechanics*, Art. 27. † Newt. *Principia*, Prop. 94, 96.

derived, are discovered by experiment, and they are wholly independent of any hypothesis respecting the manner in which the sensation is produced *.

(3.) Mr. Roemer, a Danish Astronomer, first discovered that light is propagated in time, and not communicated instantaneously from the luminous body to the eye. The discovery was made by observing that the eclipses of Jupiter's Satellites happen sooner, when he is in opposition, and later when he is in conjunction, than they ought to do according to calculations made on supposition that he is at his mean distance from the earth. To reconcile this difference between the observations and calculations, it is necessary to allow about 8′ for the time in which light passes over a radius of the earth's orbit; and the truth of the supposition is fully confirmed by Dr. Bradley's discovery of an apparent change of place in the fixed stars, which arises from the progressive motion of light, combined with the motion of the earth in it's orbit †.

(4.) Though we are, in many respects, ignorant of the nature of light, we know that it consists of distinct and independent parts.

For, it may be stopped one moment, and the next suffered to proceed; or a portion may be stopped, whilst the rest of the light is suffered to go on.

(5.) Def. The least portion of light, which may be stopped alone, or propagated alone, or do or suffer any thing which the rest of the light doth not or suffers not, is called a *Ray of Light*.

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* See *Horsley's Newt.* vol. IV. p. 305.

† The velocity of light, determined by these different observations, is nearly the same, and about 195,000 miles per second. Hence we conclude that the velocity of light is uniform; and that direct and reflected rays move at the same rate.

Rays of light are represented by lines, drawn in the directions in which the particles move.

(6.) Def. Whatever affords a passage to the rays of light is called a *Medium*; as glass, water, air, &c. and in this sense, a vacuum is called a medium.

(7.) Def. The *density* of light is measured by the number of parts, or particles uniformly diffused over a *given* surface.

Cor. If the surface be not given, the density varies as the number of particles directly, and inversely as the area over which they are uniformly diffused.

(8.) Rays of light are not lines of *contiguous* particles.

For, rays proceed from every visible point in the universe to every other point; and, in their progress, pass freely through torrents of light issuing in all directions from different suns, and different systems; but were the particles in each ray contiguous, one ray could not cross another without producing some confusion and irregularity in each; and thus vision would be rendered indistinct and precarious. Neither is such contiguity of the particles of light necessary to produce constant vision; for, if a burning coal be made to describe a circle, with a sufficient velocity, the whole circumference appears luminous; which shews that the impression made by the light upon the sensorium, when the coal is in any one point of the circumference, remains till the coal returns again to the same point*.

* It is observed that if the revolution of the coal be performed in 7‴, the whole circle appears luminous; that is, if the particles succeed each other at an interval which does not exceed that time, constant vision is produced: and since light passes over rather more than 22,000 miles in 7‴, if the distance of the particles in a ray be not

(9.) There is something extremely subtile in the nature of light; and it's properties can with difficulty be explained, either on the supposition of it's materiality, or on that of it's being only a quality of an elastic medium. The facility and regularity with which it is transmitted through bodies of considerable density, cannot be accounted for on either hypothesis. If it consist of particles of matter, which is much the more probable supposition, their minuteness greatly exceeds the limits of our faculties, even the power of human imagination. Notwithstanding the astonishing velocity of these particles (Art. 3), their momentum is not so great as to discompose the delicate texture of the eye; and when they are collected in the focus of a powerful burning glass, it seems doubtful, whether they are capable of communicating motion to the thinnest lamina of metal that can be exposed to their impact.

PROP. I.

(10.) *A ray of light, whilst it continues in the same uniform medium*, proceeds in a straight line.*

For, objects cannot be seen through bent tubes; and the shadows of bodies are terminated by straight lines. Also, the conclusions, drawn from calculations made on this supposition, are found by experience to be true.

(11.) COR. Hence it follows, that the density of light varies inversely as the square of the distance from a luminous point; supposing no particles to be stopped in their progress.

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greater than 22,000 miles, they are sufficiently near to answer the purposes of constant vision. Sir Isaac Newton supposes the impression to continue about one second of time. See *Optics*, Qu. 16.

* In speaking of a medium, we always suppose it to be uniform, unless the contrary be expressed.

For, if the point from which the light proceeds be considered as the common center of two spherical surfaces, the same particles, which are uniformly diffused over the first, will afterwards be diffused, in the same manner, over the latter; and since the density of light varies, in general, as the number of particles directly, and inversely as the space over which they are uniformly diffused (Art. 7), in this case, it varies inversely as the space over which they are diffused, because the number of particles is the same; therefore the density at the first surface : the density at the latter :: the area of the latter surface : the area of the former, that is, :: the square of the distance in the latter case : the square of the distance in the former*.

(12.) Def. When a ray of light, incident upon any surface, is turned back into the medium in which it was moving, it is said to be *reflected.*

(13.) Def. When a ray of light passes out of one medium into another, and has it's direction changed at the common surface of the two mediums, it is said to be *refracted.*

(14.) Def. The angle contained between the incident ray and the perpendicular to the reflecting, or refracting surface at the point of incidence, is called the *angle of incidence.*

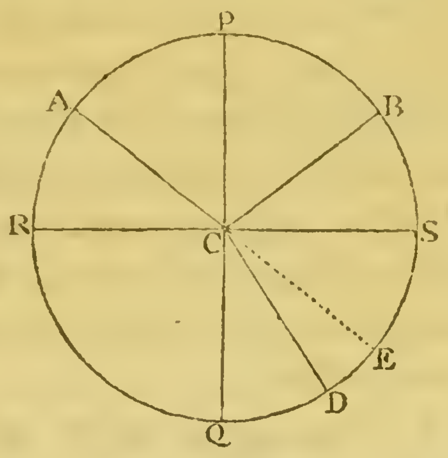
(15.) Def. The angle contained between the reflected ray and the perpendicular to the reflecting surface at the point of incidence, is called *the angle of reflection.*

(16.) Def. The angle contained between the refracted ray and the perpendicular to the refracting surface, at the point of incidence, is called *the angle of refraction.*

* *Fluxions,* page 91.

(17.) Def. The angle contained between the incident ray produced and the reflected or refracted ray, is called *the angle of deviation.*

If *RS* represent the reflecting surface, *AC* a ray incident upon it, *CB* the reflected ray, and *PCQ* be



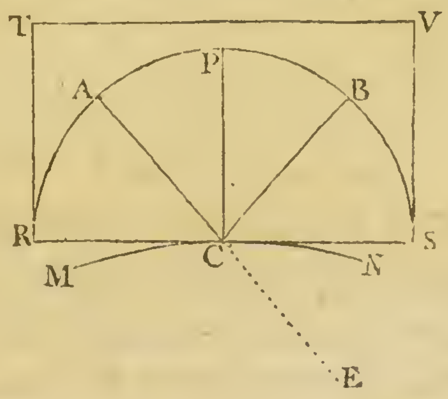
drawn, through *C*, perpendicular to *RS*, and *AC* be produced to *E*; then *ACP* is the angle of incidence, *PCB* the angle of reflection, and *BCE* the angle of deviation.

If *RS* be a refracting surface, and *CD* the refracted ray; then *QCD* is the angle of refraction, and *ECD* the angle of deviation.

Prop. II.

(18.) *The angles of incidence and reflection are in the same plane, and they are equal to each other.*

Let a ray of light *AC*, admitted through a small hole



into a dark chamber, be incident upon the reflecting

surface RS at the point C; and let CB be the reflected ray; draw CP perpendicular to the reflector. Then, if the plane surface of a board TS be made to coincide with CA and CP, the reflected ray CB is found also to coincide with the plane TS; or the angles of incidence and reflection are in the same plane.

Again, if from C as a center, with any radius CA, the circle RPS be described, the arc AP is found to be equal to the arc PB; therefore the angle of incidence, ACP, is equal to the angle of reflection, BCP.

The angles of incidence and reflection are also found to be equal, and in the same plane, when rays are reflected at a curve surface.

(19.) Cor. 1. The angles ACR, BCS, which are the complements of the angles of incidence and reflection, are also equal.

(20.) Cor. 2. If BC be the incident ray, CA will be the reflected ray. For, the angle PCA is equal to the angle PCB, and in the same plane; therefore CA is the reflected ray.

(21.) Cor. 3. If the ray PC be incident perpendicularly upon the reflecting surface, it will be reflected in the perpendicular CP.

(22.) Cor. 4. If AC be produced to E, the angle BCE, which measures the deviation of the ray AC from it's original course, is $180° - \angle ACB$; or $180° - 2 \angle$ of incidence.

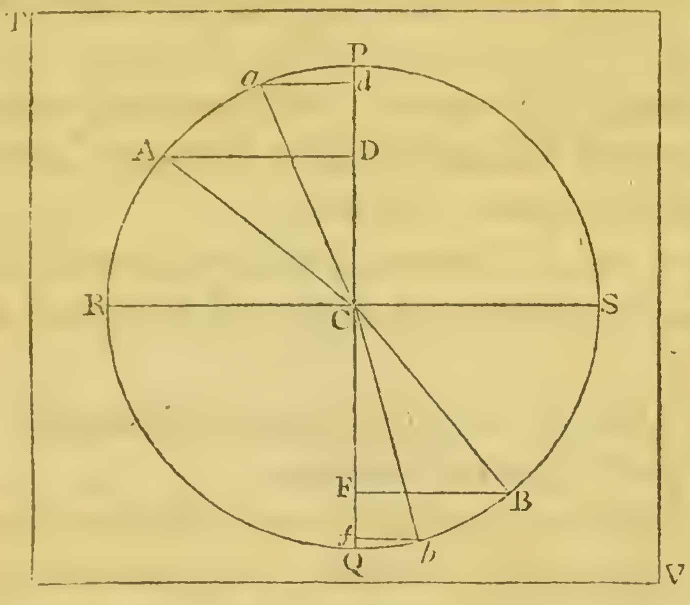
(23.) Cor. 5. A ray of light will be reflected at a curve surface, in the same manner as at a plane which touches the curve at the point of incidence.

For, the angle of incidence, and consequently the angle of reflection is the same, whether we suppose the reflection to take place at the curve, or at the plane.

Prop. III.

(24.) *The angles of incidence and refraction are in the same plane; and, whilst the mediums are the same, the sine of the angle of incidence is to the sine of the angle of refraction, in a given ratio**.

Upon the plane surface of a board *TV*, with the center *C* and any radius *CA*, describe a circle *PRQ*,



draw the diameters *RS*, *PQ* at right angles to each other, and immerse the board into a vessel of water, in such a manner that *PQ* may be perpendicular to, and *RS* coincide with the surface of the water. Then, if a ray of light, admitted through a small hole into a dark chamber, be incident upon the surface *RS* in the direction *AC*, coincident with the plane of the board, *CB*, the direction of the refracted ray, is found to coin-

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* The latter part of this proposition is only to be understood of rays of the same kind. At present it is not necessary to take into consideration the unequal refrangibility of differently coloured rays.

The sines of the angles of incidence and refraction are usually, for the sake of conciseness, called *the sines of incidence and refraction.*

cide with that plane; that is, the angles of incidence and refraction are in the same plane.

Also, if AD and BF be drawn at right angles to PQ, they are the sines of incidence and refraction, to the radius CA; and it is found that AD has to BF the same ratio, whatever be the inclination of the incident ray to the refracting surface. That is, if aC be any other incident ray, Cb the refracted ray, ad and bf the sines of incidence and refraction, then $AD : BF :: ad : bf$.

The ratio of the sines of incidence and refraction is the same, and the angles are in the same plane, when the refracting surface is curved.

(25.) COR. 1. Hence, if the angles of incidence of two rays be equal, the angles of refraction are also equal.

(26.) COR. 2. As the angle of incidence increases, the angle of refraction increases.

For, if the angle of incidence, which is always less than a right angle, increase, it's sine increases; and therefore the sine of refraction, which bears an invariable ratio to the sine of incidence, increases; and consequently the angle of refraction increases.

(27.) COR. 3. When the angle of incidence vanishes, the angle of refraction vanishes also. In this case the ray suffers no refraction.

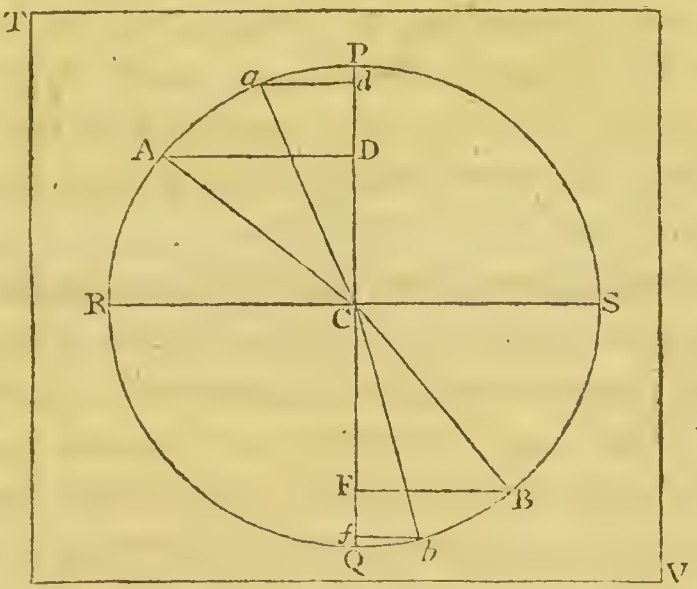
(28.) COR. 4. A ray of light is refracted at a curve surface, in the same manner as at a plane which touches the curve at the point of incidence, if the refracting power of the mediums be the same.

For, the angle of incidence, and consequently the angle of refraction is the same, whether we suppose the refraction to take place at the curve, or at the plane, supposing them to be mediums of the same kind.

Prop. IV.

(29.) *If a ray* AC *be refracted at the surface* RS *in the direction* CB, *then a ray* BC, *coming the contrary way, will be refracted in the direction* CA.

The construction being made as before, let a small object be placed upon the board at *B*; and when the board is immersed perpendicularly in water, till *RS* coincides with the surface, the object *B* will be seen



from *A*, in the direction *AC*; and since the motion of light, in the same medium, is rectilinear (Art. 10), the ray, by which the object is seen, is incident at *C*, and refracted in the direction *CA*.

(30.) Cor. 1. The angle of deviation of the ray *AC*, is equal to the angle of deviation of the ray *BC*, which is incident in the contrary direction.

(31.) Cor. 2. When a ray of light passes out of air into water, the sine of incidence : the sine of refraction :: 4 : 3; consequently, when a ray passes out of water

into air, the sine of incidence : the sine of refraction :: 3 : 4 *.

In the same manner, out of air into glass, the sine of incidence : the sine of refraction :: 3 : 2 ; therefore out of glass into air, the sine of incidence : the sine of refraction :: 2 : 3 *.

SCHOLIUM.

(32.) The preceding propositions, which are usually called the Laws of Reflection and Refraction, are the principles upon which the theory of vision is founded. They were discovered, and their truth has been established by repeated experiments, made expressly for this purpose; and it is also confirmed by the constant agreement of the conclusions derived from them, with each other, and with experience.

The experiments here described, are rather chosen with a view to give a clear illustration of these laws, than for practical application in proving, exactly, their truth. The laws of reflection are indeed easily established, but to determine with accuracy the proportion of the sines of incidence and refraction, in different cases, recourse must be had to expedients which cannot, in this place, be explained. The learner, when a little farther advanced in the subject, may consult on this head, Sir I. Newton's *Optics*, Sect. 2. and the *Encyclopædia Britannica*, Art. *Telescopes*, p. 356.

(33.) When a ray of light passes out of a rarer medium into a denser, that is, out of one which is

* These numbers do not express the exact proportions, as will be seen hereafter; but they are sufficiently accurate for our present purpose.

specifically lighter, into one which is specifically heavier, it is, in general, turned towards the perpendicular; and the contrary.

Though this is not universally the case *, yet in the subsequent part of the work, when we have occasion to speak of a denser medium, we shall always suppose it to have a greater refracting power.

(34.) When light is reflected or refracted at a polished surface, the motion of the general body of the rays is conformable to the laws above laid down; some are indeed thrown to the eye in whatever situation it is placed; and consequently, a part of the light is dispersed, in all directions, by the irregularity of the medium upon which it is incident. This dispersion is, however, much less than would necessarily be produced, were the rays reflected or refracted by the solid parts of bodies; because, the most polished surfaces, that human art can produce, must have inequalities incomparably greater than the particles of light. This, and other considerations, led Sir I. Newton to conclude that these effects are produced by some power, or medium, which is evenly diffused over every surface, and extends to a small, though finite distance from it †.

That bodies do act upon light before it comes into contact with them, is manifest from the shadows of hairs, small needles, &c. which are much larger than they ought to be, on supposition that rays pass by them in straight lines. In order to examine this phenomenon more minutely, Sir I. Newton admitted a small beam of light into a darkened chamber, and causing it to pass near the edge of a sharp knife, he

* *Newt. Optics*, Book II. Part 3. Prop. 10.

† Ibid. Book II. Prop. 8.

found that the rays were turned considerably from their rectilinear course, and that those rays were more inflected, or bent, which passed at a less distance from the edge, than those which were more remote. He also observed, that some of the rays were turned towards the edge, and others from it; so that rays of light, at different distances from the surfaces of bodies, are apparently acted upon by two different powers, one of which attracts, and the other repels them *.

The laws, according to which these powers vary, have not yet been discovered; but supposing the effects produced by them, at the same distance from a given surface, to be always the same, Sir I. Newton has shewn, that if small bodies were reflected and refracted by them, the angles of incidence and reflection would be equal; and the sines of the angles of incidence and refraction, in a given ratio to each other †. These conclusions leave us little room to doubt but that reflection and refraction are produced by such powers; and they afford some ground for presuming that the particles of light are material.

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* *Newt. Optics,* Book III.

† *Principia,* Prop. 94, 96. *Optics,* Book I, Part 1, Prop. 6.

SECT. II.

ON THE REFLECTION OF RAYS AT PLANE AND SPHERICAL SURFACES.

DEFINITIONS.

Art. (35.) BY a *pencil of rays* we understand a number of rays taken collectively, and distinct from the rest.

These pencils consist either of *parallel, converging,* or *diverging* rays.

Converging rays are such as approach to each other in their progress, and, if not intercepted, at length meet.

Diverging rays are such as recede from each other, and whose directions meet if produced backwards.

(36.) The *focus* of a pencil of rays is that point towards which they converge, or from which they diverge.

If the rays in a pencil, after reflection, or refraction, do not meet exactly in the same point, the pencil must be diminished; and the focus is the limit of the intersections of the extreme rays, when they approach nearer and nearer to each other, and at length coincide. In this case, the focus is usually called the *Geometrical Focus.*

The focus is *real*, when the rays actually meet in that point; and *imaginary*, or *virtual*, when their directions must be produced to meet.

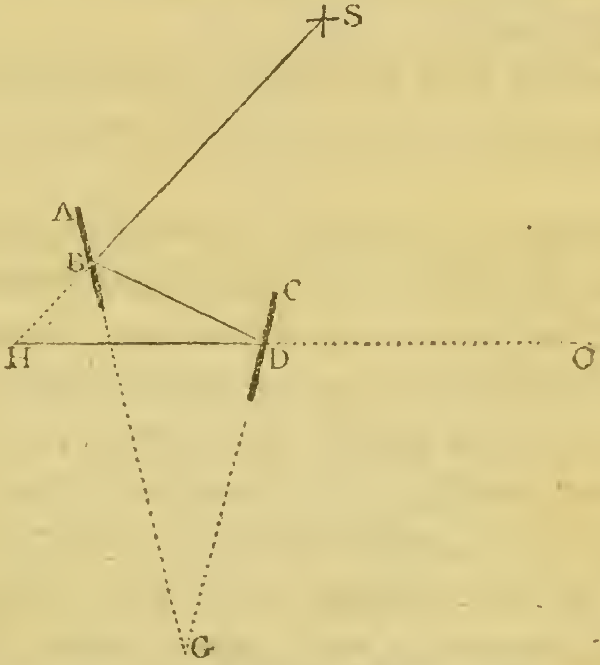
(37.) The *axis* of a pencil is that ray which is incident perpendicularly upon the reflecting or refracting surface.

(38.) The *principal focus* of a reflector, or refractor, is the geometrical focus of parallel rays incident nearly perpendicularly upon it.

PROP. V.

(39.) *If a ray of light be reflected once by each of two plane surfaces, and in a plane which is perpendicular to their common intersection, the angle contained between the first and last directions of the ray, is equal to twice the angle at which the reflectors are inclined to each other.*

Let AB, CD be two plane reflectors, inclined at the angle AGD; SB, BD, DH, the course of a ray



reflected by them. Produce HD to O, and SB till it meets DH in H. Then, because the $\angle HBG =$ the $\angle$

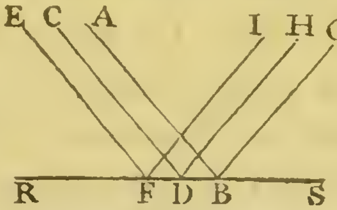
$ABS =$ the $\angle DBG$ (Art. 19), the whole angle $DBH = 2 \angle DBG$. In the same manner, the $\angle BDO = 2 \angle BDC$. And since the $\angle BGD =$ the $\angle BDC -$ the $\angle DBG$*, we have $2 \angle BGD = 2 \angle BDC - 2 \angle DBG =$ the $\angle BDO -$ the $\angle DBH =$ the $\angle BHD$*.

Prop. VI.

(40.) *Parallel rays, reflected at a plane surface, continue parallel.*

Case 1. When the angles of incidence are in the *same* plane.

Let RS be the reflecting surface; AB, CD the

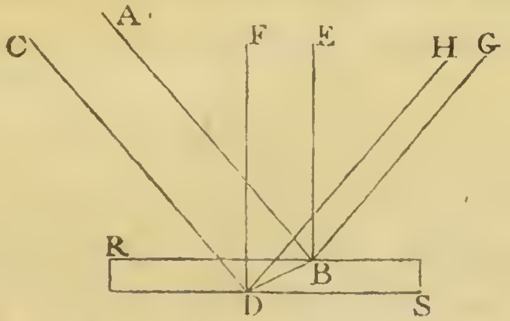


incident, BG, DH the reflected rays.

Then the $\angle ABR =$ the $\angle GBS$, and the $\angle CDR =$ the $\angle HDS$ (Art. 19); but, since AB and CD are parallel, the $\angle ABR =$ the $\angle CDR$; therefore the $\angle GBS =$ the $\angle HDS$, and BG, DH are parallel (Euc. 28. 1).

Case 2. When the angles of incidence are in *different* planes.

Let AB, CD be the incident rays; BE, DF per-



pendiculars to the reflecting surface at the points of

* *Euclid*, 32. 1.

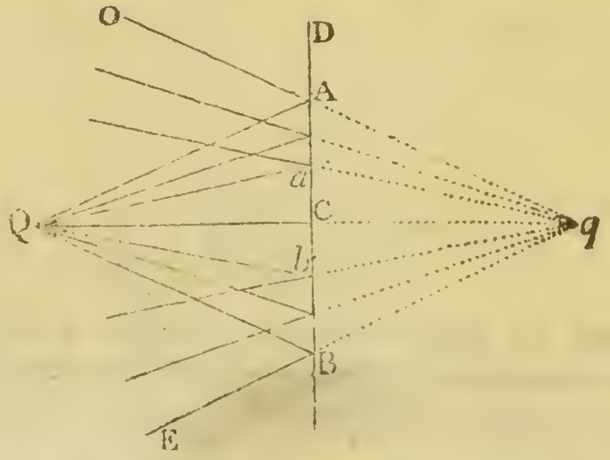
incidence; join BD; and let AB be reflected in the direction BG; also, let DH be the intersection of the planes CDF, GBD.

Then, since BE, DF, which are perpendicular to the same plane, are parallel (Euc. 6. 11), and AB, CD are parallel, by the supposition, the angles of incidence ABE, CDF are equal (Euc. 10. 11); therefore the angles of reflection are equal. Again, since EB and FD are parallel, as also AB and CD, the planes ABG, CDH are parallel (Euc. 15. 11), and they are intersected by the plane $GBDH$; consequently, DH is parallel to BG (Euc. 16. 11); therefore the angles EBG, FDH are equal (Euc. 10. 11); but the angle EBG is the angle of reflection of the ray AB; therefore the angle FDH is equal to the angle of reflection of the ray CD; and since DH is in the plane CDF, CD is reflected in the direction DH (Art. 18), which has before been shewn to be parallel to BG.

Prop. VII.

(41.) *If diverging or converging rays be reflected at a plane surface, the foci of incident and reflected rays are on contrary sides of the reflector, and equally distant from it.*

Let QAB be a pencil of rays diverging from Q,



and incident upon the plane reflector ACB; draw QC

perpendicular to the surface; then will QC be reflected in the direction CQ (Art. 21). Let QA be any other ray; and since a perpendicular to the surface at A, is in the same plane with QC and QA (Euc 6. and 7. 11), QA will be reflected in this plane (Art. 18). Produce CA to D, and make the angle DAO equal to the angle QAC, then will AO be the reflected ray (Art. 19). Produce OA, QC till they meet in q. Then, since the $\angle\, qAC =$ the $\angle\, OAD =$ the $\angle\, QAC$, and also the $\angle\, qCA =$ the $\angle\, QCA$, and the side CA is common to the two triangles QCA, CAq, the side QC is equal to Cq. In the same manner it may be shewn, that every other reflected ray in the pencil, will, if produced backwards, meet the axis in q; that is, the rays, after reflection, diverge from the focus q.

If $OABE$ be a pencil of rays converging to q, they will, after reflection at the surface ACB, converge to Q (Art. 20); therefore, in this case also, the foci of incident and reflected rays are on contrary sides of the reflector, and equally distant from it.

(42.) Cor. 1. The divergency, or convergency of rays, is not altered by reflection at a plane surface.

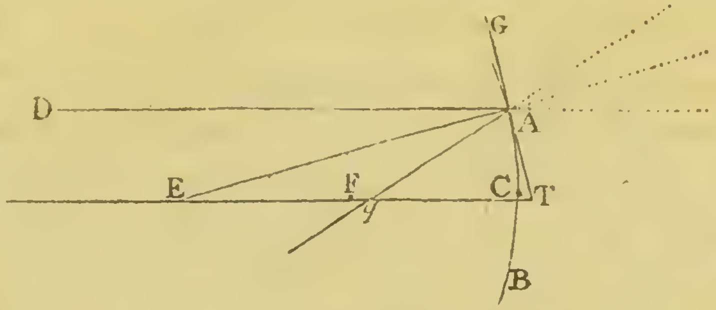
(43.) Cor. 2. In the triangles QAC, CAq, Aq is equal to QA; if therefore any reflected ray AO be produced backwards to q, making $Aq = AQ$, q is the focus of reflected rays.

(44.) Cor. 3. If the incident rays QA, Qa, be parallel, or the distance of Q from the reflector be increased without limit with respect to Aa, the distance of q is increased without limit, or the reflected rays are parallel (See Art. 40).

Prop. VIII.

(45.) *If parallel rays be incident nearly perpendicularly upon a spherical reflector, the geometrical focus of reflected rays is the middle point in the axis between the surface and center.*

Let ACB be a spherical reflector whose center is E; DA, EC, two rays of a parallel pencil incident upon it,



of which, EC passes through the center, and is therefore reflected in the direction CE; join EA, and in the plane $DACE$, make the angle EAq equal to the angle DAE, and DA will be reflected in the direction Aq (Art. 18); draw GAT, in the same plane, touching the reflector in A, and let it meet EC produced in T. Then, since the $\angle EAq =$ the $\angle DAE =$ the $\angle AEq$ (Euc. 29. 1), $Eq = Aq$; also, the $\angle qAT =$ the $\angle DAG$ (Art. 19) $=$ the $\angle ATq$ (Euc. 29. 1); therefore $Aq = qT$; consequently $Eq = qT$; that is, q bisects ET the secant of the arc AC. Now let DA approach to EC, and the arc AC will decrease, and it's secant, at length, become equal to the radius; consequently the limit of the intersections of Aq and CE is F, the middle point between E and C*.

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* It is manifest that the rays, incident nearly perpendicularly, do not meet accurately in F; but when the arc is small in comparison of the radius, in all calculations, made for the construction

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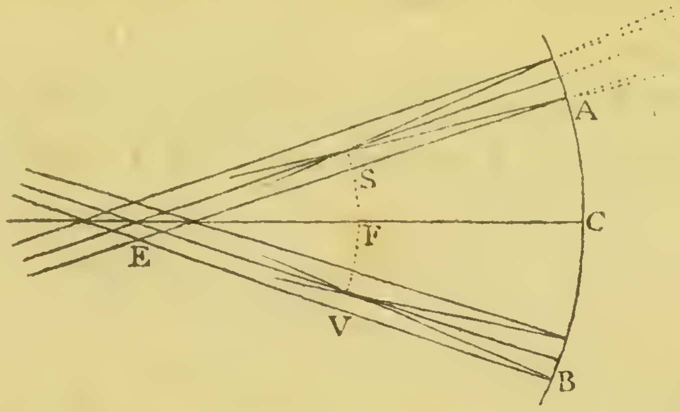
If the rays be incident upon the convex side of the reflector, the reflected rays must be produced backwards to meet the axis; and in this case, F, the middle point between E and C, may be shewn to be the limit of the intersections of CE and Aq, as before.

(46.) Cor. 1. As the arc AC decreases, Eq, or Fq, decreases. Thus, when AC is 60°, $Eq = EC$; and when AC is 45°, Aq is perpendicular to EC, and $Eq : EC :: 1 : \sqrt{2}$.

(47.) Cor. 2. If different pencils of parallel rays be respectively incident, nearly perpendicularly, upon the



reflector, the foci of reflected rays will lie in the spherical surface SFV, whose center is E and radius EF.

(48.) Cor. 3. If the axes, EA, EC, EB, of these pencils, lie in the same plane, the foci will lie in the circular arc SFV.

(49.) Cor. 4. If any point S, in the arc SFV, whose radius EF is one half of EC, be the focus of a

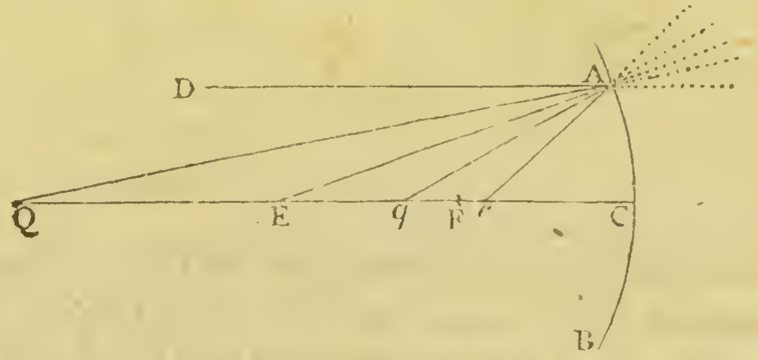
of optical instruments, F and q may be considered as coincident. Thus, if the arc AC be 20′, and the radius be divided into 100,000 equal parts, Fq is less than one of those parts; and all the rays which are incident upon the surface generated by the revolution of the arc AC about the axis EC, after reflection, cut the axis between F and q.

pencil of rays incident nearly perpendicularly upon the reflector, these rays will be reflected parallel to each other, and to EA the axis of that pencil (Art. 20).

Prop. IX.

(50.) *When diverging or converging rays are incident nearly perpendicularly upon a spherical reflector, the distance of the focus of incident rays from the principal focus, measured along the axis of the pencil, is to the distance of the principal focus from the center, as this distance is to the distance of the principal focus from the geometrical focus of reflected rays.*

Let ACB be the spherical reflector, whose center is E; Q the focus of incident rays; QA, QC two rays

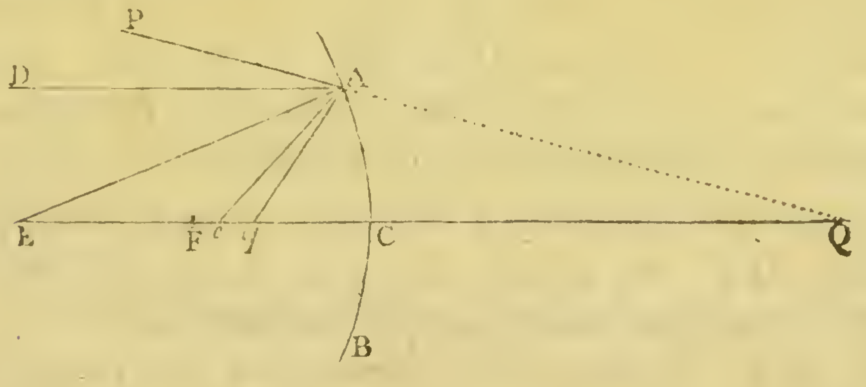


of the pencil, of which QC passes through the center E, and is therefore reflected in the direction CQ; join EA; and, in the plane $QACE$, make the angle EAq equal to the angle EAQ; then the ray QA will be reflected in the direction Aq.

Draw DA parallel to QC, and make the angle EAe equal to the angle EAD; bisect EC in F. Then, since the $\angle DAE =$ the $\angle EAe$, and the $\angle QAE =$ the $\angle EAq$, the $\angle DAQ$, or it's equal AQe, is equal to the $\angle eAq$; also, the $\angle qeA$ is common to the two triangles AQe, Aqe; therefore they are similar, and

$Qe : eA :: eA : eq$; or, since $eA = eE$ (Art. 45), $Qe : eE :: eE : eq$. Now let the arc AC be diminished without limit, or the ray QA be incident nearly perpendicularly, then e coincides with F (Art. 45); and the limit of the intersections, of CQ and Aq, is determined by the proportion $QF : FE :: FE : Fq$*.

The preceding figure is constructed for the case in which diverging rays are incident upon a concave spherical surface, and the same demonstration is applicable when the incident rays converge, as is represented in the annexed figure.



If the lines DA, QA, EA, eA, qA be produced, the figures serve for those cases in which the rays are incident upon the convex surface.

(51.) Cor. 1. If q be the focus of incident, Q will be the focus of reflected rays (Art. 20); and Q and q are called *conjugate* foci.

(52.) Cor. 2. If the distance QF be very great when compared with FE, Fq is very small when compared

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* When QA is incident *nearly* perpendicularly upon the reflector, q may be considered as coincident with that point which is determined by the proportion $QF : FE :: FE : Fq$ (See note page 20); and all other rays, incident nearly perpendicularly, will, after reflection, cut the axis in the same point; therefore that point is the focus of reflected rays.

with it. Thus, if the rays diverge from a point in the sun's disc, and fall upon a reflector whose radius does not exceed a few feet, F and q may, for all practical purposes, be considered as coincident.

(53.) COR. 3. When Q coincides with E, all the rays are incident perpendicularly upon the reflector, and therefore they are reflected perpendicularly (Art. 21), or q coincides with E.

(54.) COR. 4. The point e bisects the secant of the arc AC (Art. 45).

(55.) COR. 5. Since $Qe : eE :: eE : eq$, by composition, or division, $Qe : QE :: eE : Eq$; alternately, $Qe : eE :: QE : Eq$; and, when QA is incident nearly perpendicularly, $QF : FE :: QE : Eq$.

(56.) COR. 6. Since EA bisects the angle QAq (or PAq), $QA : Aq :: QE : Eq$ (Euc. 3. 6); and, when QA is incident nearly perpendicularly, $QC : Cq :: QE : Eq$. That is, the distances of the conjugate foci from the center, are proportional to their distances from the surface.

(57.) COR. 7. Since $QE : Eq :: QF : FE$ (Art. 55), and $QE : Eq :: QC : Cq$ (Art. 56), we have, ultimately, $QF : FE :: QC : Cq$.

(58.) COR. 8. As the arc AC decreases, Eq, the distance of the intersection of the reflected ray and the axis from the center, decreases; unless Q coincide with E, or lie between E and e.

For, $Qe : eE :: QE : Eq$ (Art. 55), and as AC decreases Ee decreases (Art. 54); therefore, when Q is in eE produced, the terms of the ratio of greater inequality, $Qe : Ee$, are equally diminished, and that ratio, or it's equal $QE : Eq$, increases (Alg. Art. 163); and, since QE is invariable, Eq decreases.

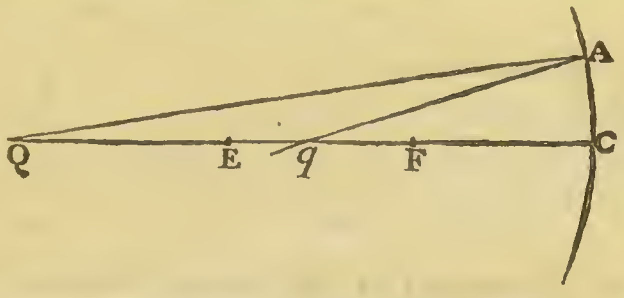
When Q is in Ee produced, as AC decreases Qe increases, and Ee decreases; therefore the ratio of $Qe : Ee$, or of $QE : Eq$ increases; and consequently, as before, Eq decreases.

But when Q lies between E and e, as AC decreases the terms of a ratio of less inequality $Qe : Ee$ are equally diminished; therefore that ratio, or it's equal $QE : Eq$, decreases (Alg. Art. 163); and since QE is invariable, Eq increases. When Q coincides with E, q also coincides with it, whatever be the magnitude of the arc AC.

Prop. X.

(59.) *The conjugate foci,* Q *and* q, *lie on the same side of the principal focus; they move in opposite directions, and meet at the center and surface of the reflector.*

Since $QF : FE :: FE : Fq$, we have $QF \times Fq = \overline{FE}|^2$; that is, Q and q are so situated that the rectangle under QF and Fq is invariable. Also, when Q coincides with E, q coincides with it (Art. 53); in



this case then, QF and Fq are measured in the same direction from F; and, since their rectangle is invariable, they must always be measured in the same direction (Alg. Art. 471).

That Q and q move in opposite directions may thus be proved: the rectangle $QF \times Fq$ is invariable; and therefore as one of these quantities increases, the

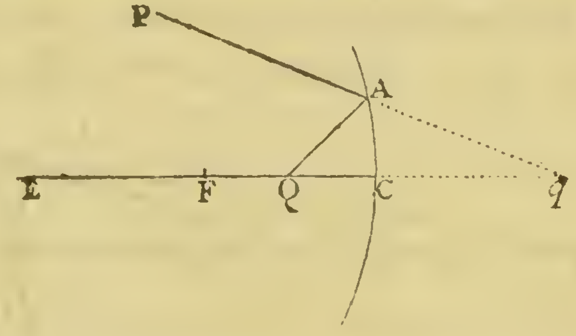
other decreases; also, Q and q lie the same way from the fixed point F; they must therefore move in opposite directions.

Having given, the place of Q, and FE the focal length of the reflector, to determine the place of the conjugate focus q, we must take $QF : FE :: FE : Fq$, and measure FQ and Fq in the same direction from F.

Thus, when Q, the focus of incident rays, is farther from the reflector than E, and on the same side of it, FQ is greater than FE, therefore FE is greater than Fq; or q, the focus of reflected rays, lies between F and E.

When Q is between E and F, q lies the other way from E; and whilst Q moves from E to F, q moves in the opposite direction from E to an infinite distance.

When Q is between F and C, QF is less than FE or FC; therefore FC is less than Fq; and, since FQ



and Fq are measured in the same direction from F, q is on the convex side of the reflector.

When Q coincides with C, QF is equal to FC; therefore FC is equal to Fq; or q coincides with C.

When converging rays are incident upon the concave surface of the reflector, QF is greater than FC; therefore FC is greater than Fq; or q lies between F and C.

(60.) Cor. 1. A concave spherical reflector lessens the divergency, or increases the convergency of all pencils of rays incident nearly perpendicularly upon it.

For, if the rays diverge from a point farther from the reflector than the principal focus, they are made to converge.

If they diverge from F, they are reflected parallel to CE.

If the focus of incidence lie between F and C, q is on the other side of the surface; or the rays diverge after reflection; and because $QF : FE :: QC : Cq$ (Art. 57), and QF is less than FE, QC is less than Cq; also, the subtense AC is common; therefore the angle contained between the incident rays QA, QC, is greater than the angle contained between the reflected rays AP, CQ; or the reflected rays diverge less than the incident rays.

If converging rays fall upon the reflector, QF (Fig. page 23) is greater than FE, therefore QC is greater than Cq; or the reflected rays converge to a focus nearer to the reflector than the focus of incident rays; and their convergency is increased.

(61.) Cor. 2. In the same manner it may be shewn, that a convex spherical reflector increases the divergency, or diminishes the convergency of all rays incident nearly perpendicularly upon it.

SCHOLIUM.

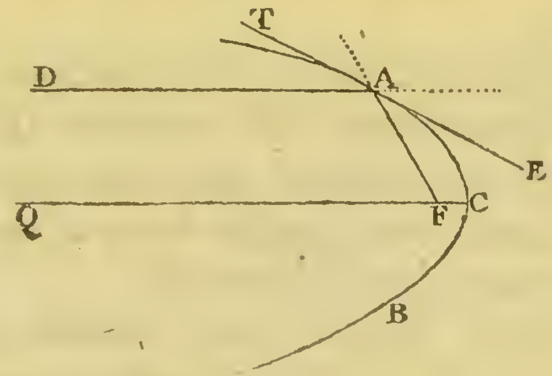
(62.) It appears from Art. 58, that unless the focus coincide with the center, a spherical reflector does not cause all the rays in a pencil either to converge or diverge accurately. This circumstance produces some confusion in vision, when these reflectors

are made use of; and, by increasing the breadth of each pencil, or, which is the same thing, by enlarging the aperture of the reflecting surface, in order to increase the quantity of light, the indistinctness thus produced is increased, as we shall have occasion to observe hereafter.

To remedy this inconvenience, it has been proposed to make use of reflecting surfaces formed by the revolution of conic sections about their axes; and it may be proper to shew that such surfaces will, in particular cases, cause rays to converge or diverge accurately.

(63.) *Parallel rays may be made to converge, or diverge accurately, by means of a parabolic reflector.*

Let ACB be a parabola, by the revolution of which about it's axis QC, a parabolic reflector is generated;



take F the focus; let DA, which is parallel to QC, be a ray of light incident upon the concave side of this reflector; and join AF. Draw TAE in the plane DAF, and touching the paraboloid in A. Then since the angle TAD is equal to the angle EAF, from the nature of the parabola, the ray DA will be reflected in the direction AF (Art. 19). In the same manner it may be shewn, that any other ray, parallel to QC, will be reflected to F; and therefore the reflected rays converge accurately to this point.

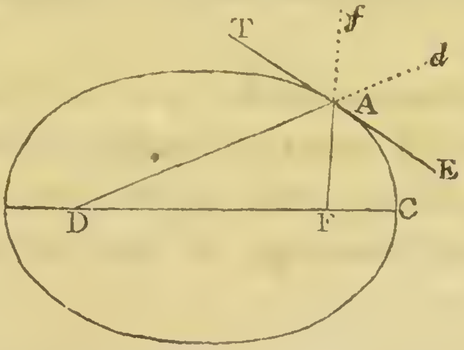
If DA, FA be produced, it is manifest that rays, incident upon the convex surface of the paraboloid, parallel to the axis, will, after reflection, diverge accurately from F.

The advantage, however, of a parabolic reflector is not so great as might, at first, be expected; for, if the pencil be inclined to the axis of the parabola, the rays will not be made to converge or diverge accurately; and the greater this inclination is, the greater will the error become.

Cor. If F be the focus of incidence, the rays will be reflected parallel to the axis.

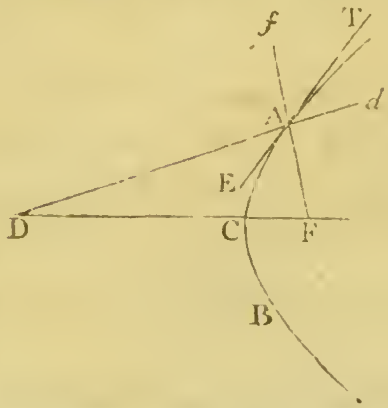
(64.) *Diverging or converging rays may be made to converge or diverge accurately, by a reflector in the form of a spheroid; and to diverge or converge accurately, by one in the form of an hyperboloid.*

Let F and D be the foci of the conic section, by the



revolution of which, about it's axis, the reflecting surface is formed; F the focus of incident rays; then will D be the focus of reflected rays.

For, let FA be an incident ray; join DA, and produce it to d; draw TAE in the plane DAF, and



touching the reflector in A; then the angle EAF is equal to the angle DAT, in the ellipse, and to dAT in the hyperbola; therefore AD is the reflected ray in the former case, and Ad in the latter; thus D is the focus of reflected rays.

If FA be produced to f, the figures serve for the cases in which rays are incident upon the convex surfaces.

We may here remark, as in the preceding article, that if rays fall upon the reflector converging to, or diverging from any other point than one of the foci, they will not converge or diverge accurately after reflection.

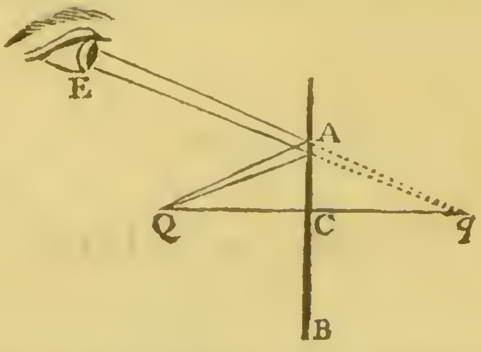
SECT. III.

ON IMAGES FORMED BY REFLECTION.

Art. (65.) THE rays of light which diverge from any point in an object, and fall upon the eye, excite a certain sensation in the mind, corresponding to which, as we know by experience, there exists an external substance in the place from which the rays proceed; and whenever the same impression is made upon the organ of vision, we expect to find a similar object, and in a similar situation. It is also evident, that if the rays belonging to any pencil, after reflection or refraction, converge to, or diverge from, a point, they will fall upon the eye, placed in a proper situation, as if they came from a real object; and therefore the mind, insensible of the change which the rays may have undergone in their passage, will conclude that there is a real object corresponding to that impression.

In some cases indeed, chiefly in reflections, the judgment is corrected by particular circumstances which have no place in naked vision, as the diminution of light, or the presence of the reflecting surface, and we are sensible of the illusion; but still the impression is made, and a representation, or *image* of the object, from which the rays originally proceeded, is formed.

Thus, the rays which diverge from Q, after reflection at the plane surface ACB, enter an eye, placed



at E, as if they came from q; or q is the image of Q.

If then the rays, which diverge from any visible point in an object, fall upon a reflecting or refracting surface, the focus of the reflected or refracted rays is the *image* of that point.

(66.) The image is said to be *real*, or *imaginary*, according as the foci of the rays by which it is formed are *real*, or *imaginary*.

(67.) The image of a physical line is determined by finding the images of all the points in the line; and of a surface, by finding the images of all the lines in the surface, or into which we may suppose the surface to be divided.

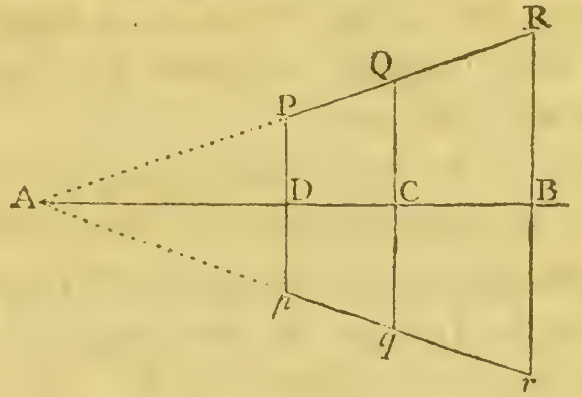
Prop. XI.

(68.) *The image of a straight line, formed by a plane reflector, is a straight line, on the other side of the reflector; the image and object are equally distant from, and equally inclined to, the reflecting plane; and they are equal to each other.*

Let PR be a straight line*, placed before the plane

* It is almost unnecessary to remind the reader, that the lines, which are considered as objects, must be physical lines, of sufficient thickness to reflect as many rays as are necessary for the purposes of vision.

reflector AB; produce RP, if necessary, till it meets the surface in A; draw RBr at right angles to AB,



and make Br equal to RB; join Ar; and from P draw PDp perpendicular to AB, meeting Ar in p; then will pr be the image of PR.

Since RBr is perpendicular to AB, and Br is equal to BR, r is the image of R (Art. 41).

Also, from the similar triangles ABR, ADP, $RB : AB :: PD : AD$; and from the similar triangles ABr, ADp, $AB : Br :: AD : Dp$; ex æquo, $RB : Br :: PD : Dp$; and since RB is equal to Br, PD is equal to Dp; or p is the image of P. In the same manner it may be shewn, that the image of every other point in PQR is the corresponding point in pqr; that is, pr is the whole image of PR.

Again, since BR is equal to Br, and AB common to the two triangles ABR, ABr, and also the angles at B are right angles, the angles of inclination RAB, BAr are equal, and AR is equal to Ar. In the same manner, AP is equal to Ap; therefore PR is equal to pr.

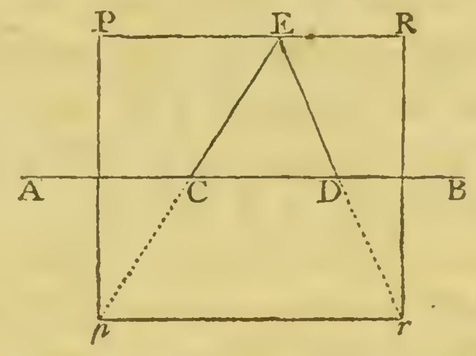
(69.) Cor. 1. If the object PR be parallel to the reflector, the image, pr will also be parallel to it.

(70.) Cor. 2. If PR be a curve, pr will be a curve, similar and equal to PR, and similarly situated on the other side of the reflector.

(71.) Cor. 3. Whatever be the form of the object, the image will be similar and equal to it. For, the image of every line in the object is an equal and corresponding line, equally inclined to, and equally distant from the reflector.

(72.) Cor. 4. If rays, converging to the several points in pr, be received upon the plane reflector, they will, after reflection, form the image PR (Art. 20).

(73.) Cor. 5. Let pr be the image of PR; and suppose an eye to be placed at E; join pE, rE,



cutting the reflector in C and D; then, considering the pupil as a point, the image will be seen in the part CD of the reflector; and it will subtend the angle CED at the eye; because all the rays enter the eye as if they came from a real object pr (Art. 65).

(74.) Cor. 6. When PR is parallel to AB, and E is situated in PR, CD is the half of pr, or PR.

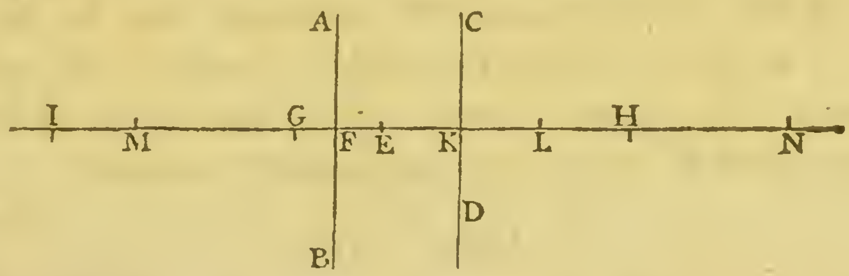
For, in this case, pr is parallel to AB (Art. 69), and therefore $CD : pr :: ED : Er :: 1 : 2$.

No rays enter the eye from any other part of the reflector.

Prop. XII.

(75.) *When an object is placed between two parallel plane reflectors, a row of images is formed which are gradually fainter as they are more remote, and at length they become invisible.*

Let AB, CD be two plane reflectors, parallel to each other; E an object placed between them; through



E draw the indefinite right line NEI perpendicular to AB, or CD. Take $FG = FE$; $KH = KG$; $FI = FH$, &c. Also, take $KL = KE$; $FM = FL$; $KN = KM$, &c.

Then, the rays which diverge from E and fall upon AB, will, after reflection, diverge from G (Art. 41); or G will be an image of E. Also, these rays, after reflection at AB, will fall upon CD as if they proceeded from a real object at G, and after reflection at CD they will diverge from H; that is, H will be an image of G; or a second image of E, &c. In the same manner, the rays which diverge from E, and fall upon CD, will form the images L, M, N, &c.

It is found by experiment, that all the light incident upon any surface, however well polished, is not regularly reflected from it. A part is dispersed in all directions (Art. 34); and a considerable portion enters the surface, and seems to be absorbed by the body.

In the passage also of light through any uniform medium, some rays are continually dispersed, or absorbed; and thus, as it is thrown backward and forward through the plate of air contained between the two reflectors *AB*, *CD*, it's quantity is diminished. On all these accounts, therefore, the succeeding images become gradually fainter, and, at length, wholly invisible.

(76.) Cor. If *E* move towards *F*, the images *G*, *H*, *I*, &c. move towards the reflectors, and *L*, *M*, *N*, &c. from them; thus the images *L* and *H*, *M* and *I*, respectively approach each other; and when *E* coincides with *F*, these pairs respectively coincide.

Prop. XIII.

(77.) *If an object be placed between two plane reflectors inclined to each other, the images formed will lie in the circumference of a circle, whose center is the intersection of the two planes, and radius the distance of the object from that intersection.*

Let *AB*, *AC* be two plane reflectors inclined at the

angle *BAC*; *E* an object placed between them. Draw

EF perpendicular to *AB*, and produce it to *G*, making *FG* = *EF*; then the rays which diverge from *E* and fall upon *AB*, will, after reflection, diverge from *G*; or *G* will be an image of *E*. From *G*, draw *GH* perpendicular to *AC*, and produce it to *I*, making *HI* = *GH*, and *I* will be a second image of *E*, &c. Again, draw *ELM* perpendicular to *AC*, and make *LM* = *EL*; also, draw *MNO* perpendicular to *AB*, and make *NO* = *MN*, &c. and *M*, *O*, &c. will be images of *E*, formed on the supposition that it is placed before *AC*. Let *K*, *V*; *P*, *Q* be the other images, determined in the same manner.

Then, since *EF* is equal to *FG*, and *AF* common to the triangles *AFG*, *AFE*, and the angles at *F* are right angles, *AG* is equal to *AE* (Euc. 4. 1). In the same manner it appears, that *AI*, *AK*, &c. *AM*, *AO*, *AP*, &c. are equal to each other, and to *AE*; that is, all the images lie in the circumference of the circle *EMIK* whose center is *A*, and radius *AE*.

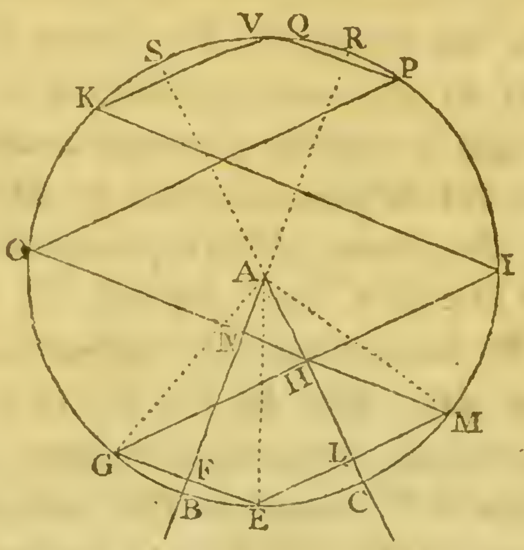
(78.) Cor. If the angle *BAC* be finite, the number of images is limited. For, *BA* and *CA* being produced to *R* and *S*, the images *Q*, *V*, will at length be formed between those points, and the rays which are reflected by either surface, diverging from any point *Q* between *S* and *R*, will not meet the other reflector; that is, no image of *Q* will be formed.

Prop. XIV.

(79.) *Having given the inclination of two plane reflectors, and the situation of an object between them, to find the number of images.*

It appears from the construction in the last proposition, that the lines *EG*, *MO*, *IK*, *PQ*, &c. are parallel, as also *EM*, *GI*, *OP*, *KV*, &c. Hence it follows, that the arcs *EG*, *MI*, *OK*, *PV*, &c. are equal;

as also, the arcs EM, GO, IP, KQ, &c. Let $BC=a$, $EB=b$, $EC=c$; then, the arc $EG=2b$; $EM=2c$;



$EO=EG+GO=EG+EM=2b+2c=2a$; $EK=EO+OK=EO+EG=2a+2b$; $EOQ=EK+KQ=EK+EM=2a+2b+2c=4a$, &c. Thus there is one series of images, formed by the reflections at AB, whose distances from E, measured along the circular arc EOR, are $2b$, $2a+2b$, $4a+2b$, ... $2na-2a+2b$ ($2na-2c$), where n is the number of images; this series will be continued as long as $2na-2a+2b$, or $2na-2c$ is less than the arc EOR, or $180°+b$; and consequently n, the number of images in this series, is that whole number which is next inferior to $\frac{180+b+2c}{2a}$, or to $\frac{180+a+c}{2a}$. There is also a second series of images, formed by reflections at the same surface, whose distances from E are $2a$, $4a$, $6a$, ... $2ma$, continued as long as $2ma$ is less than $180+b$, and therefore m, the number of these images, is that whole number which is next inferior to $\frac{180+b}{2a}$.

In the same manner, the number of images formed by reflections at the surface AC, is found by taking the whole numbers next inferior to $\frac{180+a+b}{2a}$, and $\frac{180+c}{2a}$.

(80.) COR. 1. If a be a measure of 180, the number of images formed will be $\frac{360}{a}$.

For, if a be contained an even number of times in 180, or $2a$ be a measure of 180, the number of images in each series is $\frac{180}{2a}$*; and the number upon the whole is $4 \times \frac{180}{2a} = \frac{360}{a}$. If a be contained an odd number of times in 180, $2a$ is a measure of $180+a$, or $180-a$; and the number of images is $\frac{180+a}{2a} + \frac{180-a}{2a}$† $+ \frac{180+a}{2a} + \frac{180-a}{2a} = \frac{360}{a}$.

(81.) COR. 2. When a is a measure of 180, two images coincide.

For, if a be contained an even number of times in 180, then the number of images in the second series, formed by reflections at the surface AB, is $\frac{180}{2a}$; and the distance EOQ, $(2ma)$, of the last image from E, is 180°. In the same manner, the distance EIV, of the last image in the second series formed by reflections

* $\frac{180+a+c}{2a} = \frac{180}{2a} + \frac{a+c}{2a}$; the latter part, $\frac{a+c}{2a}$, being less than unity, is neglected.

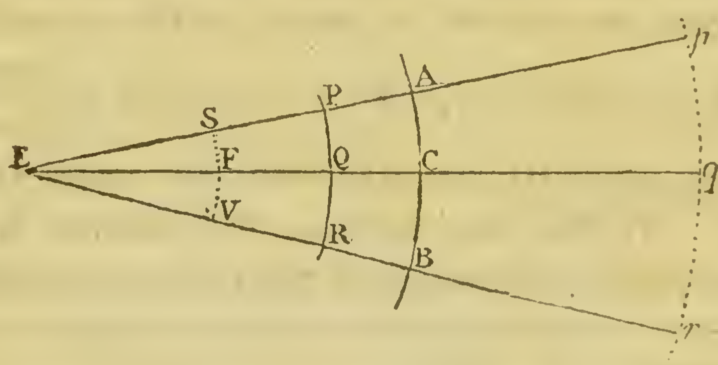
† $\frac{180+b}{2a} = \frac{180-a+a+b}{2a} = \frac{180-a}{2a} + \frac{a+b}{2a}$; the latter part, $\frac{a+b}{2a}$, being less than unity, is neglected.

at AC, is $180°$; therefore the two images, Q and V, coincide in EA produced. If a be contained an odd number of times in 180, then the number of images, in the first series, formed by reflections at AB, is $\frac{180+a}{2a}$; and the distance EOK, of the last of these images from E, is $\frac{180+a}{2a} \times 2a - 2c$; or $180°+a-2c$. Also, the distance EMP, of the last image in the first series formed by reflections at AC, is $180°+a-2b$; therefore $EOK+EMP = 360°+2a-2c-2b = 360°$; that is, K and P coincide.

Prop. XV.

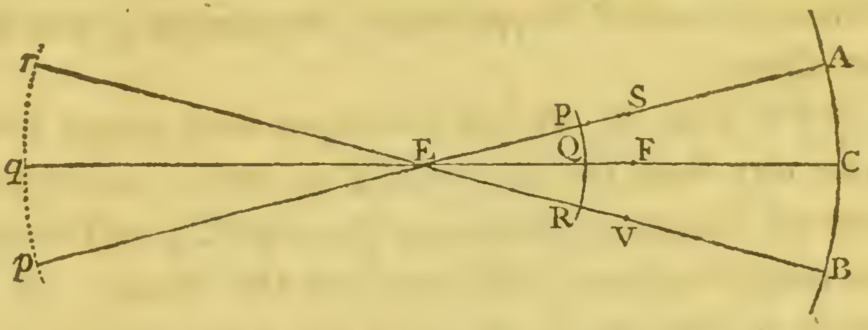
(82.) *If the object placed before a spherical reflector be a circular arc concentric with it, the image will also be a circular arc concentric with, and similar to the object.*

Let ACB be the spherical reflector, PQR the circular arc; E their common center. Take any points P, Q, R, in the object; join EP, EQ, ER, and



produce these lines, if necessary; bisect EC in F, and take $FQ : FE :: FE : Fq$, measuring Fq and FQ in the same direction from F; with the center E and radii Eq, EF, describe the circular arcs pqr, SFV,

cutting EP, ER, or those lines produced, in p, r, and



S, V; then will pqr be the image of PQR.

For, since $FQ : FE :: FE : Fq$, and FQ and Fq are measured in the same direction from F, q is the image of Q (Art. 50). Also, since EP, is equal to EQ, ES to EF, and Ep to Eq, SP is equal to FQ, and Sp to Fq; therefore $SP : SE :: SE : Sp$, and since S is the principal focus of rays incident parallel to EA, and SP and Sp are measured in the same direction from S, p is the image of P. In the same manner it may be proved, that the image of every other point in PQR, is the corresponding point in pqr; that is, pqr is the image of PQR.

Again, the image and object, since their extremities are determined by straight lines which pass through the center E, subtend the same, or equal angles at that center; therefore they are similar arcs.

(83.) Cor. 1. In the same manner it appears that, if pqr be the object, PQR will be it's image; and if rays are incident the contrary way, converging to the several points in PQR, pqr will be the image of PQR.

(84.) Cor. 2. Because similar arcs are proportional to their radii, $PR : pr :: QE : Eq$.

(85.) Cor. 3. Since $QF : FE :: QE : Eq$ (Art. 55), $PR : pr :: QF : FE$.

(86.) Cor. 4. If the object be placed any where between E and C, QF is less than FE; therefore the

object is less than the image. If it be placed in EC produced, or in CE produced, the image is less than the object.

(87.) Cor. 5. When the object and image lie the same way from the center, if P be moved successively through the several points in the object, p will move in the same direction, and trace out the image; or, the points in the image will have the same relative situation that the corresponding points in the object have; that is, the image will be erect. But if P and p be on contrary sides of E, they will move in opposite directions; or the image will be inverted.

(88.) Cor. 6. When the object is in FC, or in FC produced, the image is erect (Art. 87). When it is in FE, or FE produced, the image is inverted (See Art. 59).

(89.) Cor. 7. Since $QE : Eq :: QC : Cq$, the object and image are in the ratio of their distances from the reflector.

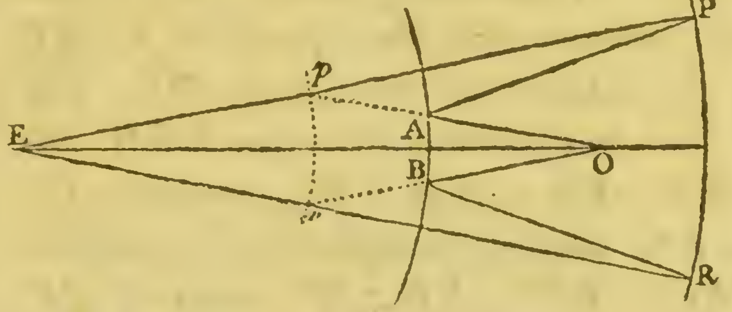
(90.) Cor. 8. If QC and qC be known, the radius of the reflector may be found. For, $QF : FE :: QC : qC$; therefore $QF \mp FE : FE :: QC \mp qC : qC$, or $QC : FE :: QC \mp qC : qC$; hence FE is known, and $2\,FE$ is the radius sought.

The upper sign is to be used when Q and q are on different sides of the reflector, and the lower, when they are on the same side.

(91.) Cor. 9. If the object be a spherical *surface*, generated by the revolution of PQR about the axis EC, the image will be a similar surface, and the magnitude of the object : the magnitude of the image :: $\overline{EQ}|^2 : \overline{Eq}|^2$.

(92.) Cor. 10. If O, the place of the eye, be given, the part of the object seen in a given portion AB, of the reflector, may be thus determined:

Join OA, OB, and produce them, if necessary, till they meet the image in p, r; then pr is the part of



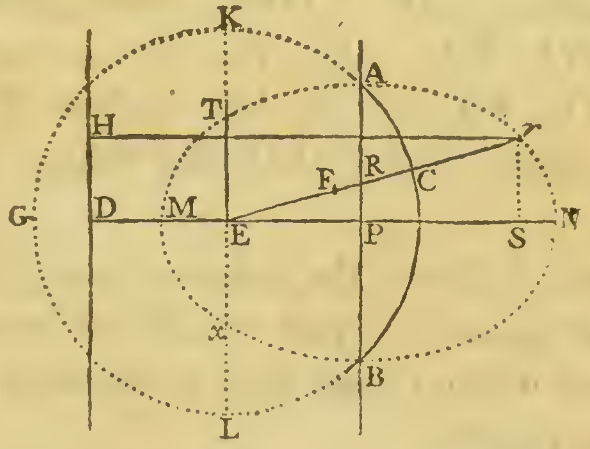
the image seen (Art. 65); join Ep, Er, and let these lines cut the object in P, R, and PR is the corresponding part of the object.

In this corollary, the pupil is supposed to be a point; and to receive the rays which are incident nearly perpendicularly upon the reflector.

Prop. XVI.

(93.) *If the object placed before a spherical reflector be a straight line, the image is a conic section.*

Let ACB be the reflector, E it's center; PR the object placed before it; through E draw $DEPN$ at



right angles to RP; take any point R in the object, join ER and produce it; bisect EC in F, and let r be

the image of R; in the line PED take ED equal to EP, and draw DH at right angles to PD; from r draw rH, rS, respectively parallel to DN, and RP.

Then, because r is the image of R, $RF : FE :: ER : Er$ (Art. 55), alternately, $RF : ER :: FE : Er$. Also, in the similar triangles EPR, ESr, $ER : Er :: EP : ES :: ED : ES$, therefore $RF : FE :: ED : ES$, and by composition, or division, $RF : RF \mp FE :: ED : ED \mp ES$; that is, $RF : ER :: ED : DS (Hr)$; consequently, $ED : Hr :: FE : Er$; alternately, $ED : FE :: Hr : Er$; that is, Hr bears an invariable ratio to Er, and therefore r is a point in the conic section whose focus is E and directrix DH.

(94.) Cor. 1. When EP is equal to EF, Hr is equal to Er, and the conic section is a *Parabola*. It is an *Ellipse* or *Hyperbola*, according as EP is *greater* or *less* than EF.

(95.) Cor. 2. When the distance of the object is so great that the rays which come from any point in it may be considered as parallel, the image is a circle whose radius is EF.

(96.) Cor. 3. If ET be drawn perpendicular to EN, when r coincides with T, Hr becomes equal to ED; and since $ED : EF :: Hr : Er$, Er becomes equal to EF. This is, ET, half the latus rectum of the conic section, is equal to EF, half the radius of the reflector.

(97.) Cor. 4. Since the radius of curvature at the vertex of the figure is equal to half the latus rectum, the curvature of the image at N is the same, wherever the object is placed.

(98.) Cor. 5. If the radius of the reflector be finite, the evanescent arc rN is equal to the ordinate rS

(*Newton*, Lem. 7); and therefore $RP : rN :: EP : ES :: EP : EN$.

Also, whilst the angle REP, which the straight line subtends at the center of the reflector is small, though finite, the arc rN will, as to sense, be a right line.

(99.) Cor. 6. In the preceding figure, where the object is supposed to lie between the principal focus and the surface ACB, TNx is the erect image of the line RP indefinitely produced both ways, and TMx it's inverted image. The part ANB is formed by reflection from the concave surface ACB; AT and Bx, by reflection from the convex surfaces AK, BL; and TMx by reflection from the concave surface KGL.

Sect. IV.

ON THE REFRACTION OF RAYS AT PLANE AND SPHERICAL SURFACES.

Prop. XVII.

Art. (100.) *When a ray of light passes out of one medium into another, as the angle of incidence increases, the angle of deviation also increases.*

Let A and B be the angles of incidence and refraction; and let sin. A : sin. B :: m : n. Then by composition and division, sin. $A +$ sin. B : sin. $A \sim$ sin. B :: $m+n : m \sim n$; but sin. $A+$sin. B : sin. $A \sim$ sin. B :: tang. $\frac{A+B}{2}$: tang. $\frac{A \sim B}{2}$*; therefore tang. $\frac{A+B}{2}$: tang. $\frac{A \sim B}{2}$ :: $m+n : m \sim n$. Now let A increase, then

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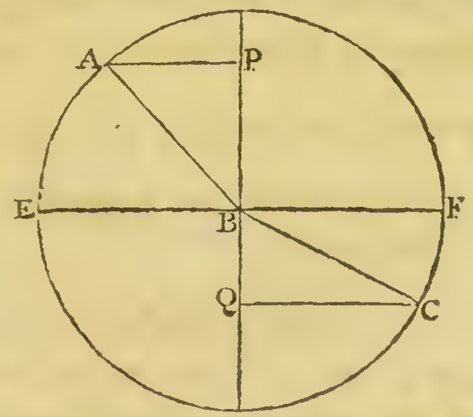
* This is deducible from the common principles of trigonometry. The sides of a triangle are proportional to the sines of the opposite angles; therefore the sum of two sides : their difference :: the sum of the sines of the opposite angles : the difference of their sines; and the sum of the sides : their difference :: the tangent of half the sum of the opposite angles : the tangent of half their difference; consequently, the sum of the sines of the angles : the difference of their sines :: the tangent of half their sum : the tangent of half their difference.

B also increases (Art. 26); therefore $\frac{A+B}{2}$ increases; and $\frac{A+B}{2}$ is less than a quadrant; therefore tang. $\frac{A+B}{2}$ increases; hence, tang. $\frac{A\sim B}{2}$, and consequently $\frac{A\sim B}{2}$, and also, $A\sim B$, the angle of deviation, increases.

Prop. XVIII.

(101.) *A ray of light cannot pass out of a denser into a rarer medium if the angle of incidence exceed a certain limit.*

Let a ray of light AB be incident upon the surface EBF of a rarer medium, and refracted in the direction BC. Through B, draw PBQ perpendicular to the surface; take BC equal to AB, and draw AP, CQ



at right angles to PQ. Then, since the ray passes out of a denser medium into a rarer, the sine of refraction is greater than the sine of incidence (Art. 33); and these sines are always in a given ratio (Art. 24); therefore the sine of refraction will become equal to the radius sooner than the sine of incidence; let the angle of incidence be increased till the sine of refraction is equal to the radius, and let the angle of incidence, and con-

sequently the sine of incidence, be farther increased; then, if the ray be refracted, the sine of refraction must also be increased, which is impossible; therefore the ray cannot, consistently with the general laws of refraction, pass out of the denser into the rarer medium, when the angle of incidence exceeds this limit.

To determine the limit, let the sine of incidence be to the sine of refraction, out of the denser medium into the rarer, :: $n : m$; and let x be the sine of incidence when the corresponding sine of refraction is r, the radius; then $n : m :: x : r$, and $x = \frac{nr}{m}$; and the sine of incidence being known, the corresponding angle may be found from a table of sines.

Thus, if the two mediums be water and air, $n : m :: 3 : 4$; therefore $x = \frac{3r}{4}$; and the angle, whose sine is $\frac{3}{4}$ of the radius, is 48°. 36′, nearly.

If the mediums be glass and air, $n : m :: 2 : 3$, and $x = \frac{2r}{3}$; in this case, the angle is 41°. 49′; and the rays which fall upon the surface at a greater angle of incidence will be reflected*.

Prop. XIX.

(102.) *When a ray of light passes through a medium contained by two parallel plane surfaces, the directions in which it is incident and emergent are parallel to each other*†.

Let *ABDC* be the medium, *PQ*, *QR*, *RS*, the

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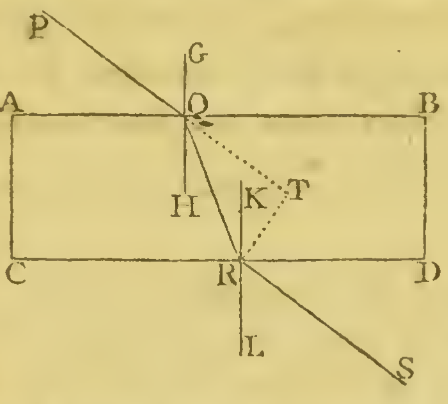
* The reflection thus made, is much stronger than can be produced by any polished metallic surface, or glass speculum.

† In this, and similar cases, the ambient medium is supposed to be uniform.

course of a ray refracted through it; GQH, KRL perpendiculars to AB and CD at the points Q and R. Now,



the effect of the refraction is the same, whether $PQRS$ be the course of the ray, or QR pass both ways out of the medium (Art. 30); let the latter supposition be made; then, since AB is parallel to CD, the alternate angles BQR, QRC are equal; and therefore their complements RQH, QRK, which are the angles of incidence, are equal; hence, the angles of refraction PQG, SRL, are equal (Art. 25); therefore the angles PQA, SRD are equal; and if to these the equal angles AQR, QRD be added, the whole angles PQR, QRS are equal, and therefore PQ, RS are parallel.

(103.) Cor. 1. Whatever be the form of the surfaces, if the planes which touch them at the points Q and R be parallel, PQ and RS will be parallel (Art. 28).

(104.) Cor. 2. If RT be drawn perpendicular to PQ produced, $QR : RT ::$ rad.: sin. RQT; wherefore, when the angle of incidence PQG, and consequently the angle of deviation RQT, is small, and the thickness of the medium also small, $PQRS$ may, without sensible error, be considered as a straight line.

Prop. XX.

(105.) *When a ray of light is refracted through two contiguous mediums, contained by parallel plane surfaces, it emerges in a direction parallel to that in which it is incident upon the first surface.*

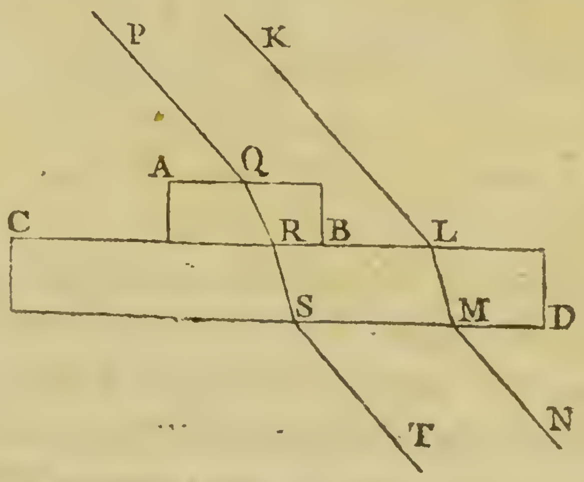
The truth of this proposition is derived from experiment.

Let a plate of glass be placed parallel to the horizon, and covered with a lamina of water; then the surface of the water will become horizontal, and therefore parallel to the surfaces of the glass plate; and if the sun's light be refracted through the two mediums, the incident and emergent rays are found to be parallel *.

Prop. XXI.

(106.) *A ray of light is as much refracted in passing through one medium into another, when they are terminated by parallel plane surfaces, as it is in passing immediately into the latter medium.*

Let *AB*, *CD* be the mediums; *PQRST* the course



of a ray refracted through them in the plane of the

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* Newt. *Lectiones Opticæ.* Par. I. Sect. II.

paper; *KLMN* the course of a ray which is incident parallel to *PQ*, and refracted, in the same plane, through the medium *CD*.

Then, since *PQ* and *ST* are parallel (Art. 105), and also *KL* and *MN* (Art. 102), and *PQ* is parallel to *KL* by the supposition, *ST* is parallel to *MN*; and if *TS* and *NM* be supposed to be the incident rays, *SR* and *ML* will be the refracted rays (Art. 29); also, since the angles of incidence, of the rays *TS*, *NM*, are equal, the angles of refraction are also equal (Art. 25); hence, the complements of these angles are equal, and *SR*, *ML* are parallel; that is, the sum, or difference, of the deviations at *Q* and *R*, is equal to the deviation at *L*.

(107.) Cor. Hence it follows, that if a ray pass through any number of mediums, contained by parallel plane surfaces, it will be as much bent from it's original course as if it passed immediately out of the first medium into the last*; and the ratio of the sine of incidence to the sine of refraction, out of the first medium into the last, is the same, whether the ray pass immediately out of the first into the last, or through any number of mediums.

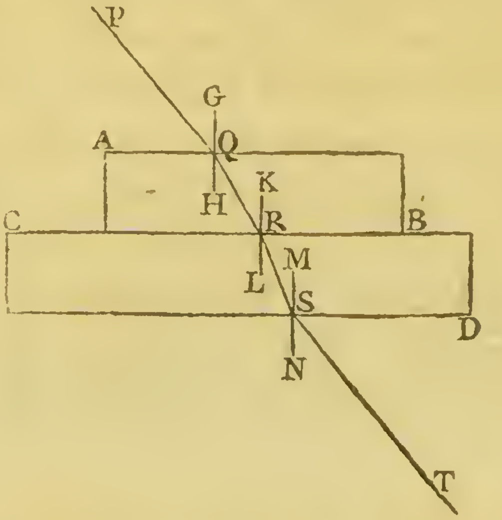
Prop. XXII.

(108.) *Having given the ratio of the sines of incidence and refraction, when a ray passes out of one medium into each of two others, to find the ratio of the sine of incidence to the sine of refraction out of one of the latter mediums into the other.*

Let *AB*, *CD* be two contiguous mediums contained by parallel plane surfaces; *PQRST* the course of a

* Newton's *Optics*, Book II. Part III. Prop. X.

ray refracted through them; through the points *Q*, *R*, *S*, draw *GQH*, *KRL*, *MSN* at right angles to the surfaces; and let $a : b$ :: sin. incidence : sin. refraction out of the surrounding medium into *AB*; $c : d$ :: sin.



incidence : sin. refraction out of the surrounding medium into *CD*.

Then, since *PQ* is parallel to *ST* (Art. 105), and *GH* to *MN*, the angles *PQG*, *NST* are equal; also, the angles *HQR*, *LRS*, are respectively equal to the angles *QRK*, *RSM*. Now, from the hypothesis, sin. *PQG* : sin. *HQR* :: $a : b$; inversely, sin. *HQR* : sin. *PQG* :: $b : a$; or

sin. *QRK* : sin. *NST* :: $b : a$; also,

sin. *NST* : sin. *RSM*(*SRL*) :: $c : d$; by composition, sin. *QRK* : sin. *SRL* :: $bc : ad$.

Ex. When a ray passes out of air into water, sin. incidence : sin. refraction :: 4 : 3 :: $a : b$; out of air into glass, sin. incidence : sin. refraction :: 3 : 2 :: $c : d$; therefore, out of water into glass, sin. incidence : sin. refraction :: $bc : ad$ :: 9 : 8.

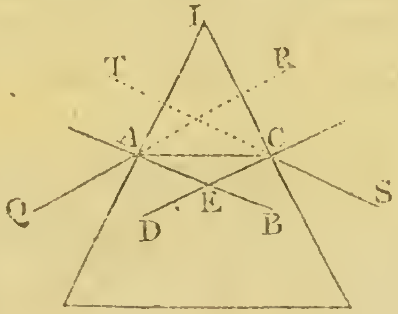
(109.) Def. A *Prism*, in optics, is a solid terminated by three rectangular parallelograms, and two similar, equal and parallel triangles.

(110.) A line drawn through the center of gravity of the prism, parallel to the intersections of the parallelograms, is called the *axis* of the prism.

Prop. XXIII.

(111.) *A ray of light which passes through a prism, in a plane perpendicular to it's axis, is turned* towards *the thicker part, or* from *it, according as the prism is* denser, *or* rarer, *than the surrounding medium.*

Let AIC represent a section of the prism, made by a plane which is perpendicular to it's axis, and therefore



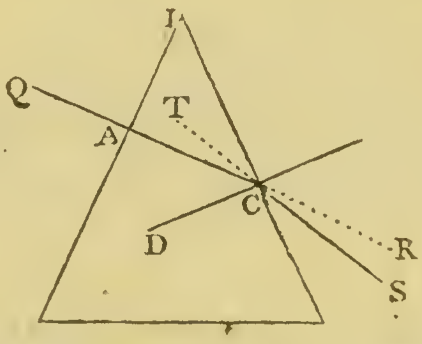
to it's surfaces (Euc. 8. and 18. 11); QA a ray incident in the plane AIC; AC, CS the course of the refracted ray, in that plane (Art. 24). Then, the effect of the refraction is the same, whether we suppose the ray to pass thus through the prism, or AC to pass both ways out of the prism (Art. 30); let this latter supposition be made, and the proposition resolves itself into the three following cases:

Case 1. When AC makes two acute angles with the sides of the prism.

Draw AB, CD at right angles to IA, IC, and let them meet in E. Then, since the $\angle CAI$ is less than the $\angle EAI$, CA is nearer to the vertex I than EA; and as they cross each other at A, CA produced is farther from the vertex than EA produced; also, the ray CA, when the prism is denser than the surrounding medium, is turned from the perpendicular; that is, from EA produced, or towards the thicker part of the prism. In the same manner it may be proved, that the ray AC is refracted at C towards the thicker part of the prism; consequently, the bending upon the whole is in that direction.

Case 2. When AC makes a right angle with one side of the prism.

Let the angle at A be a right angle; then, since there is no refraction at A (Art. 27), the whole bend-

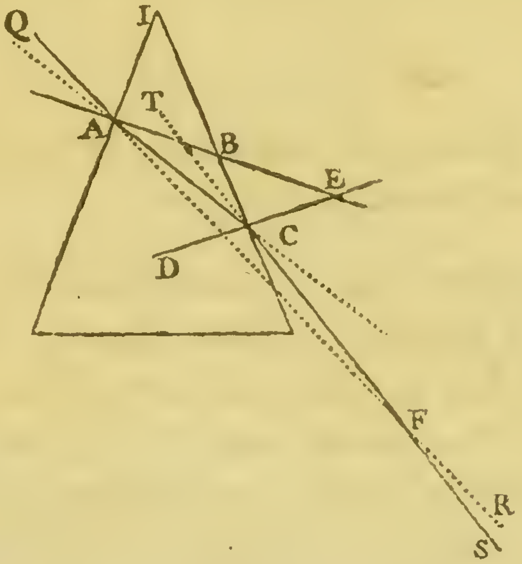


ing is at C, which may be shewn, as before, to be towards the thicker part of the prism.

Case 3. When AC makes an obtuse angle with one side of the prism.

Let the angle IAC be obtuse, and the construction being made as before, CA lies nearer to the base of the prism than EA; and CA produced lies farther from the base than EA produced; also, the ray CA, in

passing from a denser medium into a rarer, is turned from the perpendicular; that is, from EA produced,



or from the base. Thus then, the refraction at A is from the thicker part of the prism, and the refraction at C, as appears from the first case, is the contrary way; therefore the refraction, upon the whole, is the difference of the refractions at A and C. Now, the angle IBA being less than a right angle, the angles BAC, ACB are together less than one right angle; to these add the right angle BCE, and the angles BAC, ACB, BCE, or BAC, ACE, are together less than two right angles; therefore AB and CE will meet, if produced, above AC (Euc. Ax. 12); let them meet in E. Then, since the exterior angle DCA, of the triangle CAE, is greater than the interior and opposite angle CAE, the angle of incidence of the ray AC is greater than the angle of incidence of the ray CA, and therefore the deviation at C is greater than the deviation at A (Art. 100); or the excess of the deviation is towards the thicker part of the prism.

In the same manner it may be proved, that a ray of light will be bent *from* the thicker part of a prism which is *rarer* than the surrounding medium.

Prop. XXIV.

(112.) *Evanescent angles are proportional to their sines, when the radius is given.*

The limiting ratio of an evanescent arc to it's sine is a ratio of equality (*Newt.* Lem. 7. Cor. 1); and since angles are proportional to the arcs which subtend them, when the radius is given, they are, in this case, also proportional to the sines of those arcs.

When the angles are small, though of finite magnitude, the same proposition is nearly true; and sufficiently accurate, if the conclusions drawn from it be considered in a practical light, and applied to the construction of optical instruments, or the explanation of the phænomena of refraction.

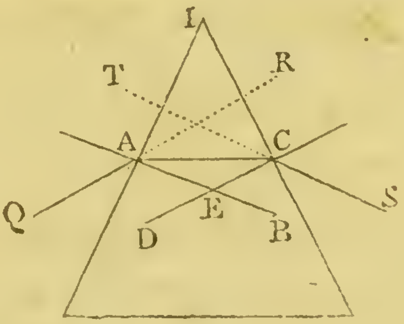
Prop. XXV.

(113.) *When a ray of light passes through a prism, in a plane which is perpendicular to it's axis, and the angles of incidence are small, the vertical angle of the prism is to the angle of deviation, as the sine of incidence, out of the prism into the ambient medium, is to the difference of the sines of incidence and refraction.*

The same construction and supposition being made as in Art. 111, the proposition will, in like manner, resolve itself into three cases.

Case 1. When *AC* makes two acute angles with the sides of the prism.

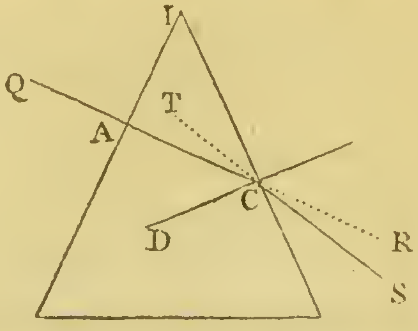
Let m : n :: sin. incidence : sin. refraction, out of the prism into the ambient medium; produce QA, SC, to R and T; then the $\angle\, CAE$ is the angle of



incidence of the ray CA, and the $\angle\, EAR$ is equal to the angle of refraction of the same ray; also the $\angle\, ECA$ is the angle of incidence, and the $\angle\, ECT$ equal to the angle of refraction of the ray AC; and since the angles of incidence, and consequently the angles of refraction are small, they are proportional to their sines (Art. 112); therefore $EAC : EAR :: m : n$; and $EAC : EAC \sim EAR\ (CAR) :: m : m \sim n$. In the same manner, $ECA : ACT :: m : m \sim n$; hence $EAC : CAR :: ECA : ACT$, and $EAC + ECA : CAR + ACT :: ECA : ACT :: m : m \sim n$ (Euc. 12. 5). Again, since the four angles of the quadrilateral figure $IAEC$ are equal to four right angles, and the angles IAE, ICE are right angles, the two angles AEC, AIC are together equal to two right angles, or to the two angles AEC, AED; consequently, the angle AIC is equal to the angle AED; and AED is equal to the sum of the angles EAC, ECA; therefore the angle AIC is equal to the sum of those angles; also, the sum of the angles CAR, ACT is, in this case, the whole deviation (Prop. 23. Case 1.); therefore, from the last proportion, AIC : the whole deviation :: $m : m \sim n$.

CASE 2. When AC is at right angles to one side of the prism.

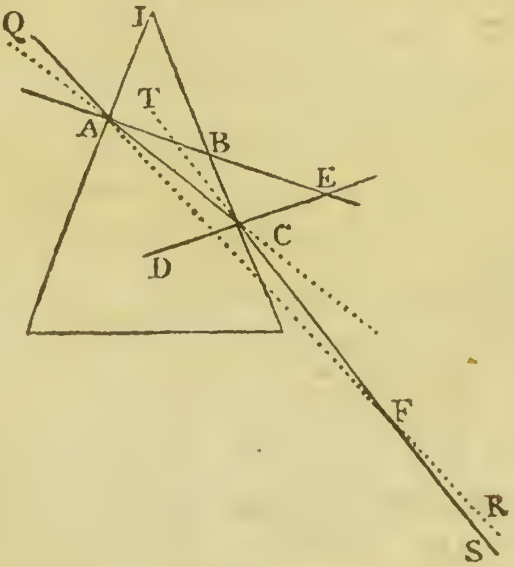
Let the angle CAI be a right angle; then the whole refraction is at C; and in this case, as before, $DCA : ACT :: m : m \sim n$. Also, since the right angle



DCI is equal to the sum of the two ACI, AIC, take away the common angle ACI, and the remaining angles DCA, AIC are equal; consequently $AIC : ACT :: m : m \sim n$.

CASE 3. When AC makes an obtuse angle with one side of the prism.

It may be shewn in the same manner as in Case 1,



that $ACD : ACT :: m : m \sim n$; and $CAE : CAF :: m : m \sim n$. Hence, $ACD - CAE : ACT - CAF ::$

$m : m \sim n$ (Euc. 19. 5.); and since ACD is the exterior angle of the triangle ACE (Prop. 23. Case 3), $ACD - CAE = CEA$; also, $ACT - CAF$ is the whole deviation; therefore CEA : the deviation :: $m : m \sim n$. Again, since the triangles AIB, BCE, have vertical angles at B, and right angles IAB, BCE, the angles AIB, BEC are equal; therefore, from the last proportion, AIB : the whole deviation :: $m : m \sim n$.

(114.) Cor. 1. It appears from the demonstration of the foregoing proposition, that when the ray makes two acute angles with the sides of the prism, the angle at the vertex is equal to the *sum* of the angles of incidence; and when it makes an obtuse angle with one side, the angle at the vertex is equal to the *difference* of the angles of incidence.

(115.) Cor. 2. Hence it follows, that the angles of incidence cannot be small, unless the angle at the vertex of the prism be also small.

Ex. 1. Let the angle at the vertex of a glass prism, placed in air, be $1°$; then $m : n :: 2 : 3$, and $m : m \sim n :: 2 : 1 :: 1° : \frac{1}{2}°$, the angle of deviation when the angles of incidence are small.

Ex. 2. Let the same prism be placed in water, then $m : n :: 8 : 9$; and $m : m \sim n :: 8 : 1 :: 1° : \frac{1}{8}°$, the angle of deviation in this case.

(116.) Cor. 3. When the angle at the vertex of the prism vanishes, or the sides become parallel, the angle of deviation also vanishes (See Art. 102.)

(117.) Cor. 4. If the quantity and direction of the deviation, and the angle at the vertex of the prism be known, the ratio of the sine of incidence to the sine of refraction may be determined.

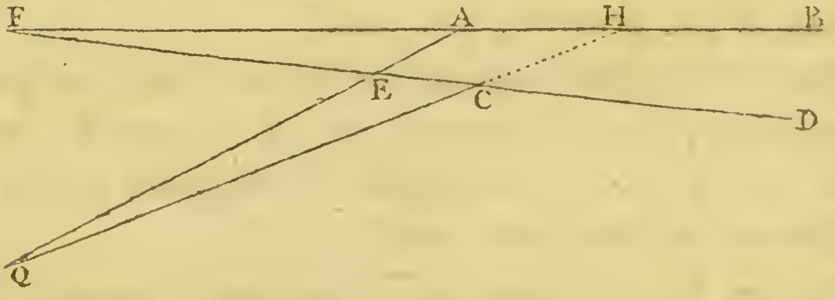
Thus, if the deviation be $\frac{1}{3}$ of the angle at the vertex, and towards the thicker part of the prism, $m : n-m :: 3 : 1$; and by composition, $m : n :: 3 : 4$.

If the deviation be towards the thinner part of the prism, $m : m-n :: 3 : 1$; therefore $m : n :: 3 : 2$. In these, as in the former cases, the angles of incidence and refraction are supposed to be proportional to their sines.

Prop. XXVI.

(118.) *When two rays are refracted through a prism, in the same plane, and the angles of incidence on each surface are small, the emergent rays are inclined to each other at an angle equal to that which is contained between the incident rays.*

Let QA, QC be the incident rays; AB, CD the directions of the refracted rays; and, if possible, let



AB, CD be parallel. Produce QC till it meets FAB in H; then since QH falls upon the parallel lines AB, ED, the angles ECQ and AHC are equal; and the exterior angle FAE, of the triangle AHQ, is greater than the interior and opposite angle AHQ; therefore it is also greater than the angle ECQ; but, because the angle at the vertex of the prism is invariable, the angles FAE, ECQ, which are equal to the angles of deviation of the rays QA, QC, are equal to each other (Art. 113), which is absurd; therefore AB, and CD

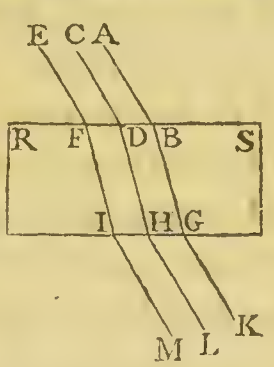
are not parallel *. Let them meet in F; then since the vertical angles FEA, QEC are equal, and also the angles FAE, ECQ, the remaining angles AFE, EQC are equal.

Prop. XXVII.

(119.) *Parallel rays, refracted at a plane surface, continue parallel* †.

Case 1. When the angles of incidence are in the same plane.

Let RS be the plane refracting surface; AB, CD the incident, BG, DH the refracted rays. Then, since AB, CD are parallel, the angles ABR, CDR are equal; and therefore the complements of these, or the angles of incidence are equal; hence, the angles of refraction are equal, and consequently the comple-



ments of the angles of refraction, that is, the angles SBG, SDH are equal; and therefore BG and DH are parallel.

Case 2. When the angles of incidence are not in the same plane.

Let AB, CD be the incident rays; EBF, KDL

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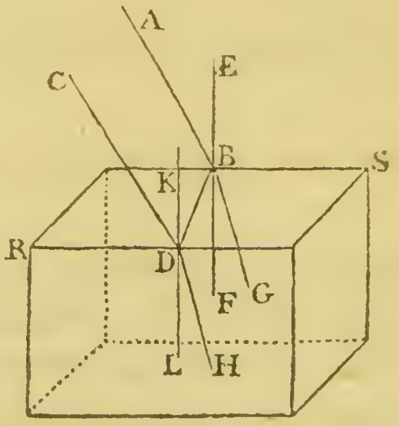
* See also Prop. 30.

† In this, and the following propositions, the rays are supposed to be equally refrangible.

perpendiculars to the refracting surface *RS*, at the points of incidence *B* and *D*; join *BD*; and let *AB* be refracted in the direction *BG*, which lies in the plane *ABF* (Art. 24); also, let *DH* be the intersection of the planes *GBD*, *CDL*.



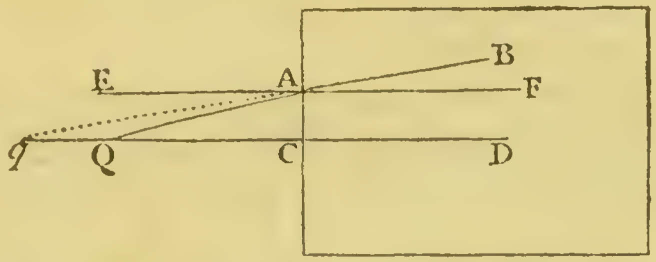
Then, since *EF*, *KL* are parallel (Euc. 6. 11), as also *AB*, *CD*, by the supposition, the angles of incidence *ABE*, *CDK* are equal (Euc. 10. 11); consequently the angles of refraction are equal. Again, because *EF* and *KL* are parallel, and also *AB* and *CD*, the planes *ABF*, *CDL* are parallel (Euc. 15. 11); and the plane *GBD* intersects them; hence it follows, that *BG* and *DH* are parallel (Euc. 16. 11); and therefore the angles *GBF* and *HDL* are equal (Euc. 10. 11); but the angle *GBF* is the angle of refraction of the ray *AB*; therefore *HDL* is equal to the angle of refraction of the ray *CD*; and since *DH* is in the plane *CDL*, *CD* is refracted in the direction *DH* (Art. 24), which has before been shewn to be parallel to *BG*.

Prop. XXVIII.

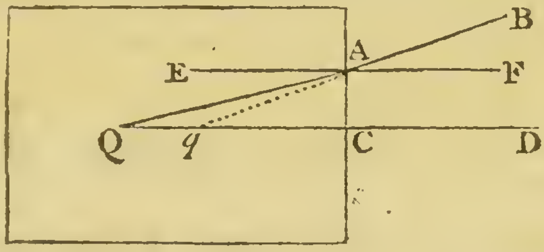
(120.) *When diverging or converging rays are incident nearly perpendicularly upon a plane refracting surface, the distance of the focus of incident rays*

from the surface, is to the distance of the geometrical focus of refracted rays from the surface, as the sine of refraction to the sine of incidence.

Let AC be the plane refracting surface; QA, QC two of a pencil of rays diverging from Q, of which QC is perpendicular to the surface, and therefore suffers no



refraction. Through A, draw EAF parallel to QC; and let QA be refracted in the direction AB; produce BA till it meets CQ, or CQ produced, in q*. Then, the $\angle EAQ$ is the angle of incidence of the ray QA, and the $\angle BAF$ the angle of refraction; also, the $\angle$



EAQ is equal to the alternate angle AQC, and the $\angle$ BAF is equal to the interior and opposite angle AqC; therefore, sin. EAQ, or sin. incidence, = sin. AQC = sin. AQq; and sin. BAF, or sin. refraction, = sin. AqC = sin. AqQ; hence, $QA : qA$ :: (sin. AqQ : sin. AQq ::) sin. refraction : sin. incidence. Now, let A approximate to C, and QA is ultimately equal to QC,

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* If they do not meet, AB coincides with AF; that is, the angle of refraction vanishes with respect to the angle of incidence; or the refracting power is infinite.

and qA to qC; therefore, the proportion becomes, $QC : qC$ :: sin. refraction : sin. incidence *.

Let q be the focus of rays incident the contrary way; then BAF is the angle of incidence, and EAQ the angle of refraction (Art. 29); and it may be proved as before, that, when Q is the limit of the intersections of the refracted rays and the axis, $qC : QC$ :: sin. refraction : sin. incidence.

Ex. 1. When diverging rays pass out of air into water, according to the hypothesis made in the proposition, $QC : qC :: 3 : 4$; and $QC : Qq :: 3 : 1$.

Ex. 2. When diverging rays pass out of water into air, $QC : qC :: 4 : 3$; and $QC : Qq :: 4 : 1$.

Ex. 3. When converging rays pass, in the same manner, out of air into glass, $QC : qC :: 2 : 3$; when they pass out of glass into air, $QC : qC :: 3 : 2$.

(121.) Cor. 1. A plane refracting surface of a denser medium, diminishes the divergency of diverging rays, and the convergency of converging rays, incident nearly perpendicularly upon it. A plane refracting surface of a rarer medium produces the contrary effect.

(122.) Cor. 2. As the point A approaches to C, q approaches to Q.

Let sin. incidence : sin. refraction :: $m : n$; then $QA : qA :: n : m$; and $QA^2 : qA^2 :: n^2 : m^2$; or $QC^2 + CA^2 : qC^2 + CA^2 :: n^2 : m^2$; hence $QC^2 + CA^2 : QC^2 \sim qC^2 :: n^2 : n^2 \sim m^2$; and since the point Q is fixed, and QC invariable, as also the ratio of $n^2 : n^2 \sim m^2$, when CA decreases, $QC^2 + CA^2$ decreases; and therefore $QC^2 \sim qC^2$ decreases; consequently Qq decreases.

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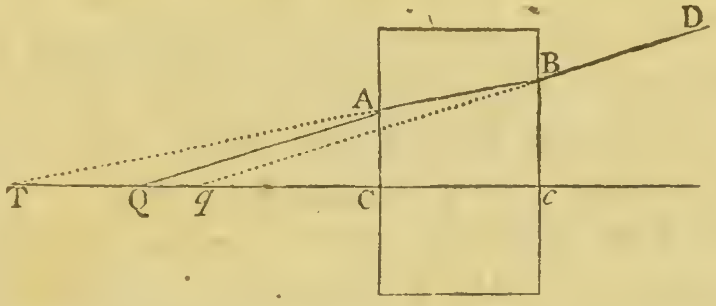
* See Note, page 20.

Prop. XXIX.

(123.) *When diverging or converging rays pass, nearly perpendicularly, through a medium contained by parallel plane surfaces, the distance of the foci of incident and emergent rays, is to the thickness of the medium, as the difference of the sines of incidence and refraction, to the sine of incidence upon the first surface.*

Let $ACcB$ be the medium, QA, QC two rays of a pencil incident upon it, of which QC is perpendicular to the surface AC, and therefore passes through the medium without suffering any refraction; let QA be refracted in the direction AB, and emergent in the



direction BD. Produce BA and DB, till they meet the axis in T and q.

Then, because QA and qB are parallel (Art. 102), $TA : AB :: TQ : Qq$ (Euc. 2. 6); and because AC is parallel to Bc, $TA : AB :: TC : Cc$; therefore $TQ : Qq :: TC : Cc$; and alternately, $TQ : TC :: Qq : Cc$. Now let A approximate to C, and T is, ultimately, the geometrical focus of the rays after the first refraction; therefore $QC : TC ::$ sin. refraction : sin. incidence (Art. 120); and by division, $QT : TC ::$ sin. incidence $\sim$ sin. refraction : sin. incidence; consequently, $Qq : Cc ::$ sin. incidence $\sim$ sin. refraction : sin. incidence.

If rays, incident the contrary way, converge to q, they will, after both refractions, converge to Q (Art. 29); therefore, as before, $Qq : Cc$ :: sin. incidence $\sim$ sin. refraction : sin. incidence on the first surface.

Ex. If the medium be glass, placed in air, $Qq : Cc :: 1 : 3$; if water, placed in air, $Qq : Cc :: 1 : 4$.

(124.) Cor. 1. When the incident rays diverge, the geometrical focus of emergent rays is nearer to c, or farther from it than the focus of incident rays, according as $ACcB$ is denser, or rarer than the ambient medium.

Let $ACcB$ be denser than the ambient medium; and let sin. incidence : sin. refraction :: $m : n$. Then, $TC : QC :: m : n$; therefore Tc has to Qc a less ratio than that of $m : n$ (Alg. Art. 162). Also, $Tc : qc :: m : n$ (Art. 120); consequently, Tc has a less ratio to Qc than to qc; or qc is less than Qc. In the same manner it appears, that if $ACcB$ be rarer than the ambient medium, qc is greater than Qc.

(125.) Cor. 2. Since CT is greater, or less than CQ, according as $ACcB$ is denser, or rarer than the ambient medium (Art. 120), it is manifest that T and q lie on opposite sides of Q.

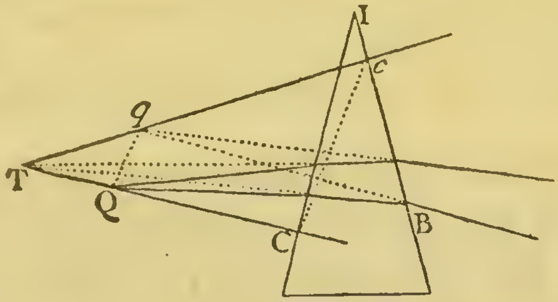
(126.) Cor. 3. When $ACcB$ is denser than the surrounding medium, as A approaches to C, T approaches to Q (Art. 122); and in consequence of this motion of T, q approaches to c. Again, as A approaches to C, B approaches to c; and on this account q approaches to T (Art. 122), or recedes from c. Thus the two motions of q are in opposite directions, and the aberration of oblique rays, from the geometrical focus, is less than when the rays are refracted at a single surface.

The same may be shewn when $ACcB$ is rarer than the ambient medium.

Prop. XXX.

(127.) *Having given the focus of incidence of a pencil of rays which passes nearly perpendicularly through the sides of a prism, and also the ratio of the sine of incidence to the sine of refraction, out of the ambient medium into the prism, to find the focus of emergent rays.*

Let CIB be the prism; Q the focus of incident rays; take $m : n$ :: sin. incidence : sin. refraction out of the ambient medium into the prism. From Q, draw QC perpendicular to IC; and in CQ, or CQ



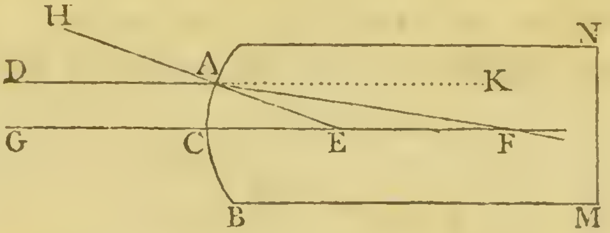
produced, take $TC : QC :: m : n$; then will T be the focus after refraction at the surface IC (Art. 120), or the focus of rays incident upon the surface IB. From T, draw Tc perpendicular to IB, and in cT, or cT produced, take $qc : Tc :: n : m$; and q will be the focus of emergent rays (Art. 120).

(128.) Cor. Since $QC : TC :: n : m :: qc : Tc$, if Cc and Qq be joined, these lines are parallel (Euc 2. 6); and therefore $Qq : Cc :: TQ : TC :: m \backsim n : m$.

Prop. XXXI.

(129.) *Parallel rays, refracted at a convex spherical surface of a denser, or a concave of a rarer medium, into which they pass, are made to converge; and refracted at a concave spherical surface of a denser, or convex of a rarer medium, they are made to diverge.*

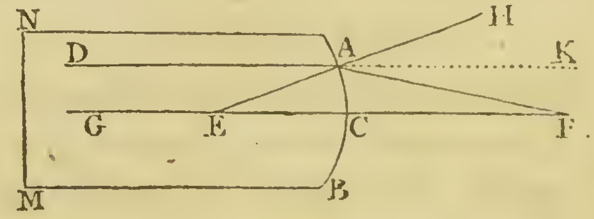
1. Let *DA*, *GC* be two rays of a parallel pencil, passing out of a rarer medium into a denser, and incident upon the convex spherical surface *ACB*, whose center is *E*. Let *GCE* pass through the center of the surface, and it suffers no refraction. Join *EA*, and produce it to *H*; also, produce *DA* to *K*; and let *DA* be refracted in the direction *AF*; then, *DAH* is the angle of incidence, and *EAF* the angle of refraction of this ray; and since it passes out of a rarer medium



into a denser, the ∠ *EAF* is less than the ∠ *HAD*, and therefore it is less than the ∠ *KAE*; add to each the ∠ *AEF*, and the two angles *FAE*, *AEF* are together less than the two *KAE*, *AEF*; and therefore they are less than two right angles (Euc. 29. 1); consequently, *AF*, and *GE*, if produced, will meet.

2. When the rays pass out of a denser medium into a rarer, and the surface of the medium into which they are refracted is spherically concave.

The construction being made as before, since the ray *DA* passes out of a denser medium into a rarer,



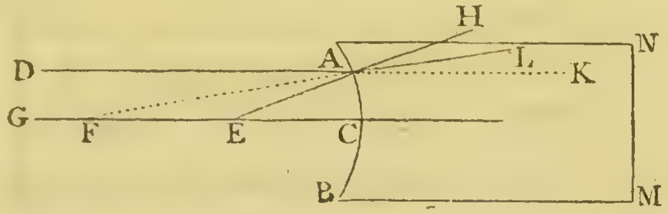
the angle of incidence *DAE*, or it's equal *AEC*, is less than the angle of refraction *HAF*; add to each

the angle EAF, and the two EAF, AEF, are together less than the two EAF, HAF; that is, they are together less than two right angles; therefore AF and EC, if produced, will meet.

3. When the rays pass out of a rarer medium into a denser, and the surface of the medium into which they are refracted is spherically concave.

The same construction being made, let DA be refracted in the direction AL, and produce LA to F.

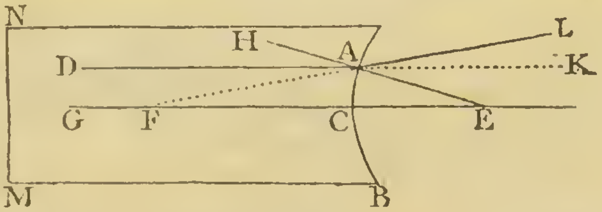
Then, since the ray DA passes out of a rarer medium into a denser, the $\angle DAE$ is greater than the $\angle HAL$,



or FAE; add to each the $\angle AEG$, and the two FAE, AEG, are together less than the two DAE, AEG; that is, they are less than two right angles; therefore AF and EG will meet.

4. When the rays pass out of a denser medium into a rarer, and the surface of the medium into which they are refracted is spherically convex.

In this case, as before, the $\angle DAH$, or it's equal AEC, is less than the $\angle EAL$; add to each the $\angle$



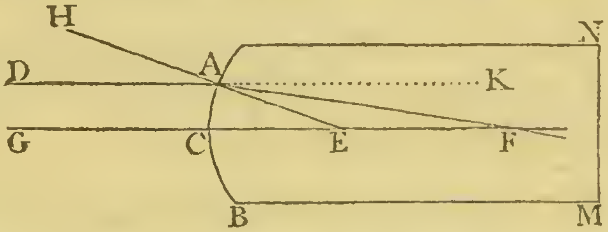
EAF, and the two EAF, AEC, are together less than the two EAF, EAL; that is, they are less than two right angles; therefore AF and EC, if produced, will meet.

In the two last cases, the refracted rays meet if produced backwards; that is, they diverge.

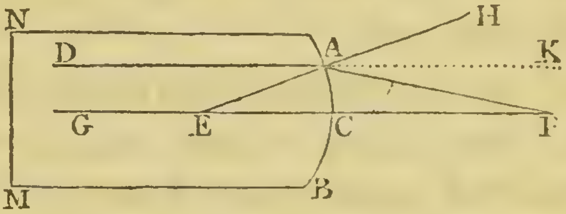
Prop. XXXII.

(130.) *When parallel rays are incident nearly perpendicularly upon a spherical refracting surface, the distance of the geometrical focus of refracted rays from the surface, is to it's distance from the center, as the sine of incidence to the sine of refraction.*

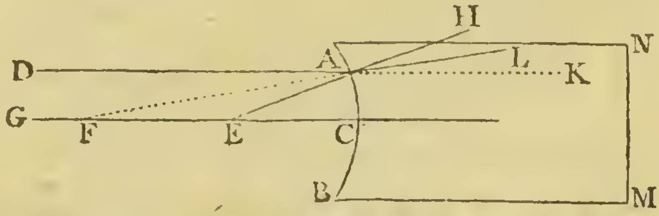
The construction being made as in the last proposition, the angle AEF, in each of the four cases, is



either equal to the angle of incidence of the ray DA, or to it's supplement; therefore sin. $\angle AEF =$ sin. in-

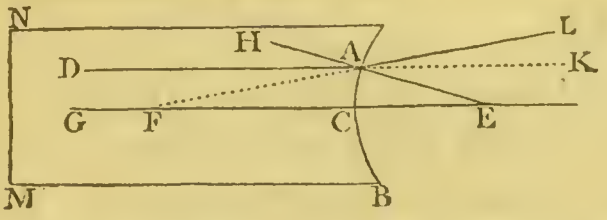


cidence. In the same manner, sin. $\angle EAF =$ sin. refraction; and since the sides of a triangle are pro-



portional to the sines of the opposite angles, $FA : FE ::$ sin. $\angle AEF$: sin. $\angle EAF$ :: sin. incidence : sin. refraction. Now let the point A approximate to C, and

FA is, ultimately, equal to *FC*; therefore, the pro-



portion becomes *FC* : *FE* :: sin. incidence : sin. refraction.

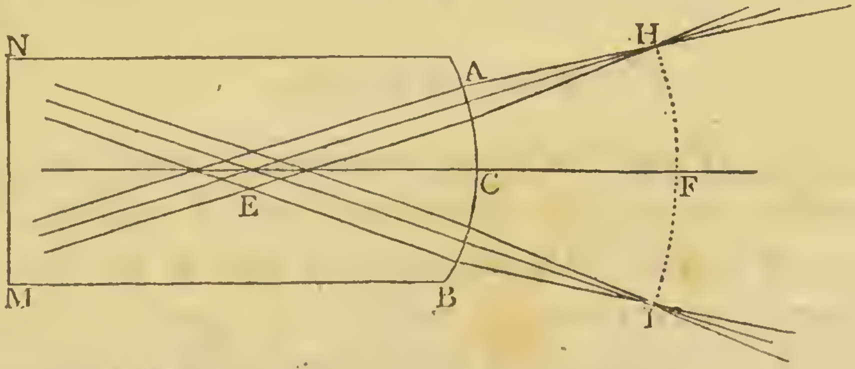
(131.) Cor. 1. Since *FC* : *FE* :: sin. incidence : sin. refraction, by division, *FC* : *CE* :: sin. incidence : sin. incidence$\sim$sin. refraction. Also, *FE* : *CE* :: sin. refraction : sin. incidence$\sim$sin. refraction.

Ex. 1. If parallel rays pass out of air into the medium *ABMN* of glass, *FC* : *FE* :: 3 : 2; also, *FC* : *EC* :: 3 : 1; and *FE* : *EC* :: 2 : 1.

Ex. 2. When the rays pass out of glass into air, *FC* : *FE* :: 2 : 3; also, *FC* : *EC* :: 2 : 1; and *FE* : *EC* :: 3 : 1.

(132.) Cor. 2. If *EC* be diminished whilst the refracting power remains unaltered, *FC* is also diminished.

(133.) Cor. 3. If the axes of several pencils of



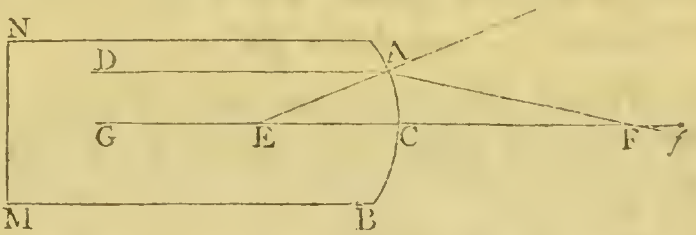
parallel rays be inclined to each other, in the same plane, the foci of refracted rays will lie in the circumference of a circle, *HFI*, whose center is *E*, and radius

EF. If the axes be in different planes, the foci will lie in the surface of a sphere, whose center is E, and radius EF.

(134.) Cor. 4. If any point H, in the arc HFI, be the focus of rays incident the contrary way, join HE, and those rays of the pencil which are incident nearly perpendicularly, will be refracted parallel to each other, and to HE (Art. 29).

(135.) Cor. 5. The distance EF, of the intersection of the refracted ray and the axis, from the center, is the greatest, when the arc AC is evanescent.

Let f be the geometrical focus; $m : n$ the ratio of the sine of incidence to the sine of refraction. Then,



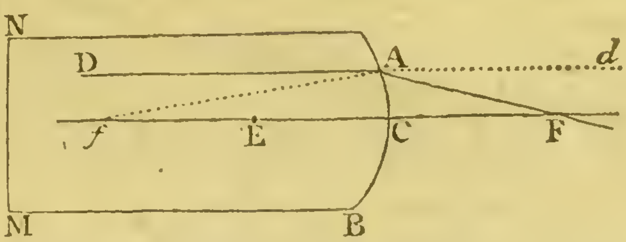
$Ef : EC :: n : m \sim n$; also, $EF : AF :: n : m$; therefore $EF : EF \sim AF :: n : m \sim n$; hence $Ef : EC :: EF : EF \sim AF$; but $EF \sim AF$ is less than EA, or EC (Euc. 20. 1); consequently, EF is less than Ef.

Prop. XXXIII.

(136.) *If parallel rays be incident, nearly perpendicularly, in opposite directions, upon a spherical refracting surface, the distance of one of the foci of refracted rays from the surface, is equal to the distance of the other from the center of the refractor.*

Let F be the focus when the rays pass out of the denser medium into the rarer, and f the focus when

they pass out of the rarer into the denser; then FC :

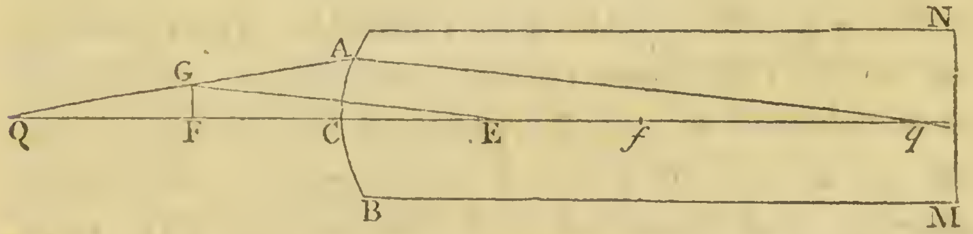


$CE :: n : m-n$ (Art. 131); also, $fE : CE :: n : m-n$; therefore $FC : CE :: fE : CE$; or $FC=fE$. By adding EC to, or subtracting it from each of these equal quantities, $FE=fC$.

Prop. XXXIV.

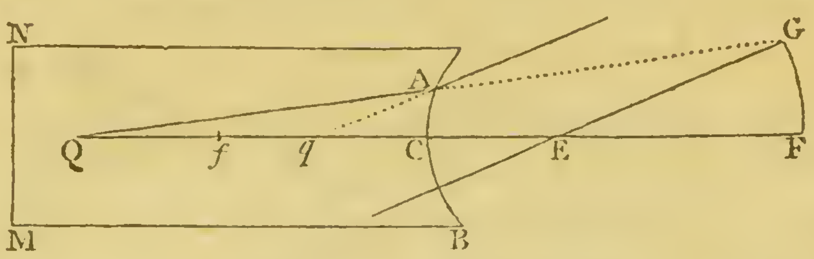
(137.) *When diverging or converging rays are incident nearly perpendicularly upon a spherical refracting surface, the distance of the focus of incident rays from the principal focus of rays coming in the contrary direction, is to it's distance from the center of the refractor, as it's distance from the surface, to it's distance from the geometrical focus of refracted rays.*

Let ACB be the refracting surface; E it's center; Q the focus of incident rays; QA and QC two rays



of the pencil, of which QCE passes through the center, and therefore suffers no refraction. Take F the principal focus of rays incident in the contrary direction, parallel to EC; and from the center E,

with the radius EF, describe the arc FG, cutting QA,



or QA produced, in G; join EG; draw Aq parallel to EG, and let it meet the axis in q.

1. When diverging rays are incident upon a convex spherical refractor of a denser medium.

The ray QA will be refracted at A, in the same manner and degree, whether it be considered as one of a pencil of rays diverging from Q, or as one of a pencil diverging from G; and, on the latter supposition, it will be refracted parallel to GE (Art. 134); therefore Aq is the refracted ray*. And since the triangles QGE, QAq are similar, $QG : QE :: QA : Qq$; let the point A approximate to C, that the ray QA may be incident nearly perpendicularly upon the refracting surface, and the point G approximates to F; therefore ultimately, $QF : QE :: QC : Qq$.

2. When diverging rays are incident upon a convex spherical surface of a rarer medium.

In this case, Q and F are on contrary sides of the refracting surface (Art. 129); and the ray QA will be refracted in the same manner and degree, whether it be considered as one of a pencil of rays diverging from Q, or, as one of a pencil converging to G; consequently, qA, produced, is the refracted ray (Art. 134).

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* This is only true when GA is incident nearly perpendicularly upon AC; and therefore the intersection of the refracted ray and the axis is only determined in that case.

Hence, as before, $QG : QE :: QA : Qq$; and ultimately, $QF : QE :: QC : Qq$. A similar proof is applicable in all the other cases*.

(138.) Cor. 1. From the same triangles, $QG : GE :: QA : Aq$; therefore, ultimately, $QF : FE :: QC : Cq$.

(139.) Cor. 2. Since GE is parallel to Aq one side of the triangle QAq, the other sides QA, Qq, or those sides produced, are cut proportionally (Euc. 2. 6); therefore $QG : GA :: QE : Eq$; and ultimately, $QF : FC :: QE : Eq$.

(140.) Cor. 3. If f be the other principal focus, and q the focus of incident rays, Q is the focus of refracted rays (Art. 29); therefore, $qf : fE :: qC : QC$ (Art. 138); invertendo, $fE : qf :: QC : Cq :: QF : FE$; hence, $QF : FE :: Ef : fq$.

Prop. XXXV.

(141.) *The distances* QF *and* Qq *must be measured in the same, or opposite directions from* Q, *according as* QC *and* QE *are measured in the same, or opposite directions from that point.*

Since $QF : QE :: QC : Qq$, we have $QF \times Qq = QE \times QC$; and, measuring these lines from Q, if the rectangles have the same sign in any one case, they will always have the same sign. Now, if QF be very great when compared with FE or EC, qf is very small (Art. 140); therefore all the lines QF, QC, QE, Qq, are measured the same way from Q, and the

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* Since the incident rays may either *converge* or *diverge*, and fall upon a *convex* or *concave* surface of a *rarer* or *denser* medium, the proposition admits of eight cases.

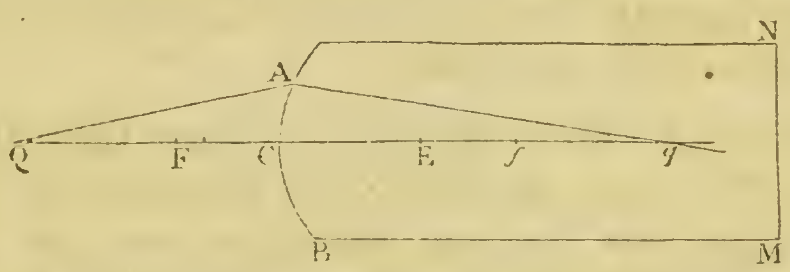
rectangles $QF \times Qq$, and $QE \times QC$, in this case, have the same sign; consequently, they will always be either both positive, or both negative; and according as QC and QE have the *same*, or *different* signs, QF and Qq must have the *same*, or *different* signs; that is, QF and Qq must be measured, from Q, in the *same*, or *different* directions, according as QC and QE are measured in the *same*, or *different* directions (Alg. Art. 471.

(142.) Nearly in the same manner, it may be shewn that QF and fq must always be measured in opposite directions from F and f.

Prop. XXXVI.

(143.) *The conjugate foci,* Q *and* q, *move in the same direction upon the indefinite line* QCq, *and they coincide at the surface and center of the refractor.*

Let the rays be incident nearly perpendicularly on ACB, a convex spherical refracting surface of a denser



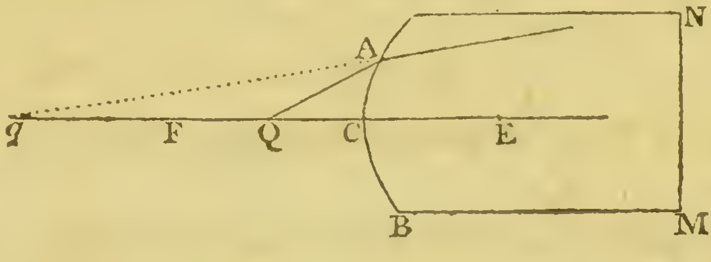
medium; take f the principal focus of rays, thus incident, and F the other principal focus.

When the incident rays are parallel, the refracted rays converge to f.

As Q approaches towards F, since $QF : FE :: QC : Cq$ (Art. 137), and the ratio of $QF : QC$ decreases (Alg. Art. 163), the ratio of $FE : Cq$ decreases; therefore Cq increases.

When Q coincides with F, the distance Cq is indefinitely great.

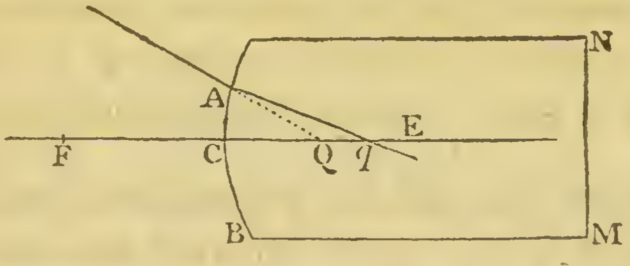
When Q is between F and C, QF and Qq are measured in the same direction from Q (Art. 141); and



as the ratio of QF to QC increases, the ratio of FE to Cq increases, or Cq decreases.

When Q coincides with C, the ratio of $QF : FE$ is finite; therefore the ratio of $QC : Cq$ is finite, and since QC vanishes, qC also vanishes; that is, q coincides with C.

When Q is between C and E, Qq must be measured from Q towards E (Art. 141); and since $QF : QE ::$

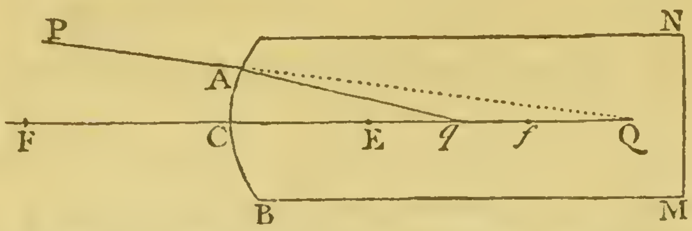


$QC : Qq$ (Art. 137), and QF is greater than QC, QE is greater than Qq; consequently, q lies between Q and E.

When Q coincides with E, since QF is equal to FE, QC is equal to Cq; or, q coincides with E.

When Q is in CE produced, Qq must be measured from Q towards C; and since $QF : QE :: QC : Qq$, and QF is greater than QC, QE is greater than Qq; that is, q lies between Q and E. Also, since QF :

$FE :: QC : Cq$ (Art. 138), and as QC or QF increases, the ratio of QF to QC decreases (Alg. Art. 162), the



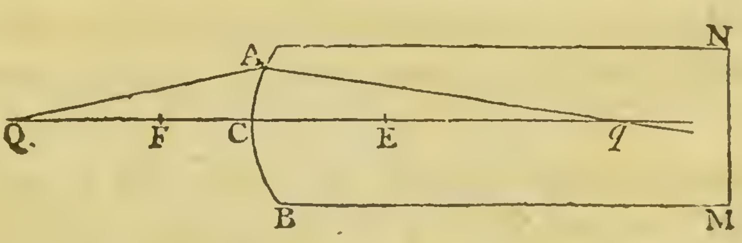
ratio of FE to Cq also decreases; that is, Cq increases.

The same demonstration, mutatis mutandis, may be applied to all the other cases.

Prop. XXXVII.

(144.) *A convex spherical refracting surface of a denser, and a concave of a rarer medium, diminish the divergency, or increase the convergency of all pencils of rays incident nearly perpendicularly upon them, unless the focus of incident rays be in the surface or center of the refractor, or between those two points; a concave spherical surface of a denser, and convex of a rarer medium, have a contrary effect.*

It appears from the last proposition, that when rays are incident upon a convex spherical surface of a denser



medium diverging from Q, a point farther from the surface than F, they are made to converge.

When the incident rays diverge from F, the refracted rays are parallel.

When Q is between F and C, the refracted rays diverge from a point which is farther from the surface than Q; therefore the divergency of the rays is diminished*.

When the incident rays converge to any point in CE produced, the refracted rays converge to a point which is nearer to the surface than the focus of incident rays; or the convergency is increased.

But when the incident rays converge to a point between C and E, the refracted rays converge to a point farther from the surface than the focus of incident rays; or the convergency is diminished.

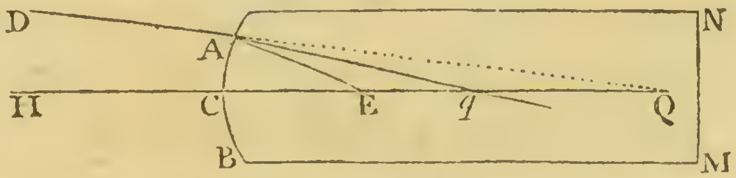
When Q coincides with C or E, the convergency, or divergency, is not altered.

In the same manner, the proposition may be proved in the other cases.

Prop. XXXVIII.

(145.) *If* E *be the center of a spherical refractor* ACB, *and, in* CE *produced,* QE *be taken to* EC :: *sin. incidence : sin. refraction, all the rays converging to* Q, *when the refracting surface is convex, and diverging from* Q, *when that surface is concave, will, after refraction, converge to, or diverge from, one point.*

When the refracting surface is convex, let DA and



HC be two rays of the pencil, of which HC passes

* See Art. 60.

through the center, and therefore suffers no refraction; and let DA be refracted in the direction Aq; join EA.

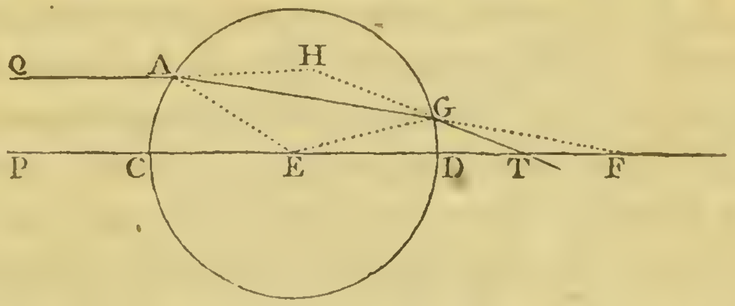
Then, since sin. incidence : sin. refraction :: QE : EA :: sin. $\angle QAE$: sin. $\angle AQE$, and the $\angle QAE$ is equal to the angle of incidence, the angle AQE is equal to the angle of refraction, that is, to the $\angle EAq$; also, the $\angle AEQ$ is common to the two triangles QAE, AqE; therefore these triangles are similar, and $QE : EA :: EA : Eq$, the three first terms of which proportion being invariable, the fourth, Eq, is also invariable; that is, all the refracted rays meet the axis in the same point.

The proposition may be proved in the same manner when the refracting surface is concave.

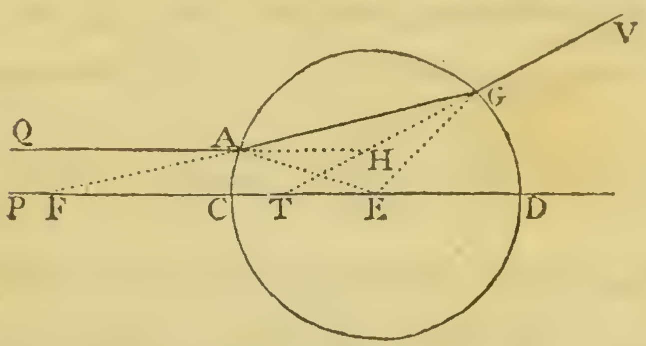
Prop. XXXIX.

(146.) *To find the principal focus of a sphere.*

Let a pencil of parallel rays be incident upon the



sphere ACD, whose center is E; and let PCE be



that ray which passes through the center, and therefore suffers no refraction at either surface (Art. 27); also,

let QA, any other ray of the pencil, be refracted in the direction AG, and emergent in the direction GT, or GV, which. produced backwards, or forwards, as the case may require, cuts the axis in T. Produce QA, and TG, or VG, till they meet in H; join EA, EG.

Then, if two rays GA, AG pass out of the sphere at A and G, the angles of incidence EAG, EGA are equal, and therefore the angles of deviation are equal; but the deviation at A is the same, whether GA or QA be the incident ray (Art. 30); consequently, when the ray QA is refracted through the sphere, the deviation at A is equal to the deviation at G; or, the $\angle HAG =$ the $\angle FGT$. Also, the $\angle HAG =$ the $\angle GFT$; therefore the $\angle GFT =$ the $\angle FGT$, and $FT = TG$. Now, let the point A approximate to C, and F is, ultimately, the principal focus of rays after the first refraction; also, the point G approximates to D*; consequently, TG is ultimately equal to TD, and therefore $FT = TD$; that is, the principal focus bisects the distance between the focus after the first refraction, and the farther extremity of the diameter in the direction of which the rays are incident.

(147.) Cor. 1. Since $2TD = FD$, and $2DE = CD$, we have $2TD \mp 2DE = FD \mp CD$; that is, $2TE = FC$.

(148.) Cor. 2. Since $FC : CE$ :: sin. incidence : sin. incidence $\sim$ sin. refraction (Art. 131), we have $2TE : CE$ :: sin. incidence : sin. incidence $\sim$ sin.

* Otherwise, F would coincide with C, which cannot be the case so long as the refracting power is finite (Art. 130).

refraction; or, $TE : CE$ :: sin. incidence : 2 sin. incidence $\sim$ 2 sin. refraction.

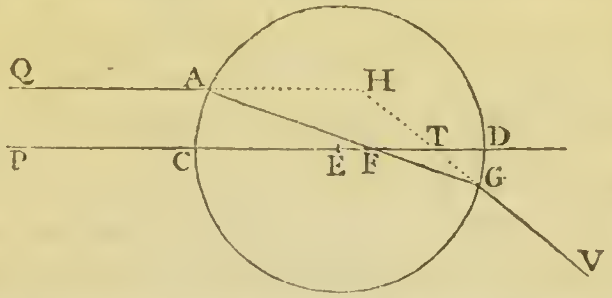
The distance TE is called the *focal length* of the sphere.

Ex. 1. If the sphere be glass, placed in air, sin. incidence : sin. refraction :: 3 : 2; therefore $TE : CE$ :: 3 : 2.

Ex. 2. If the sphere be water, placed in air, sin. incidence : sin. refraction :: 4 : 3; and $TE : CE$ :: 4 : 2 :: 2 : 1.

Ex. 3. If sin. incidence : sin. refraction :: 2 : 1, $TE : CE$ :: 2 : 2; or T coincides with D.

Ex. 4. If the sine of incidence be to the sine of refraction in a greater ratio than that of 2 : 1, T falls



within the sphere; and the rays, after the second refraction, diverge from T.

(149.) Cor. 3. If the axes of different pencils of parallel rays, incident upon the sphere, lie in the same plane, the foci will lie in the circumference of a circle whose center is E, and radius ET.

(150.) Cor. 4. If T be the focus of rays incident nearly perpendicularly upon the sphere, in the contrary direction, these rays will be refracted parallel to each other, and to TE.

(151.) Cor. 5. If the radius of the sphere, and it's focal length be known, the ratio of the sine of incidence to the sine of refraction may be determined.

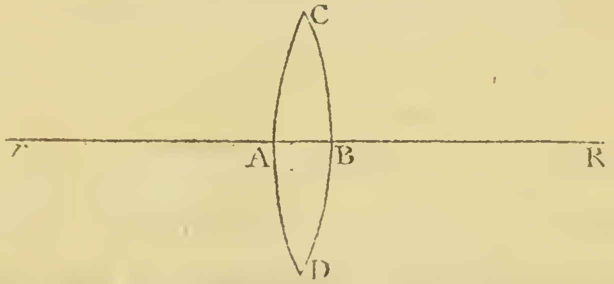
Let $m : n$ :: sin. incidence : sin. refraction; then, $2TE : EC :: m : m \sim n$; therefore $2TE : 2TE \mp EC :: m : n$; where the negative sign is to be used when the rays converge after the first refraction, and the positive sign, when they diverge.

(152.) Def. A *Lens* is a thin piece of glass, or other transparent substance, whose surfaces are either both spherical, or one plane and the other spherical.

This definition comprises the six following sorts of lenses: the *double convex*, the *double concave*, the *plano convex*, the *plano concave*, the *meniscus*, and the *concavo-convex* lens.

1. A *double convex lens* is bounded by two convex spherical surfaces.

Let R and r be the centers of two circular arcs CAD, CBD which are concave towards each other,



and which meet in C and D; join Rr, and suppose the figure CD to revolve about Rr as an axis; the solid, thus generated, is called a *double convex lens*.

The line Rr is called the *axis* of the lens.

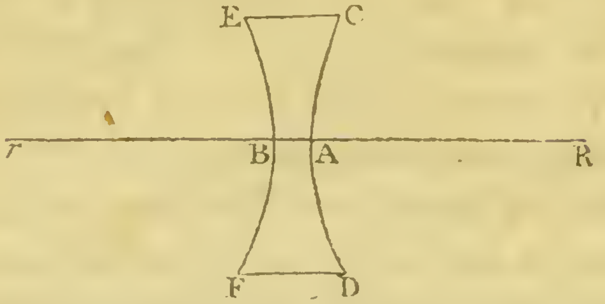
The figure $CADB$ represents a section of the lens, made by a plane which passes through the axis.

If CD be joined, this line is called the *diameter*, or *linear aperture* of the lens.

2. A *double concave lens* has both it's surfaces concave.

Let R and r be the centers of two circular arcs

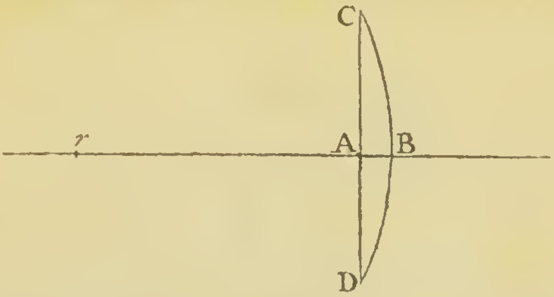
which are convex towards each other; join *Rr*; draw



EC, *FD* parallel to *Rr*, and equally distant from it; then the solid generated by the revolution of the figure *ECDF*, about the axis *Rr*, is called *a double concave lens.*

3. A *plano-convex lens* is bounded by a plane, and a convex spherical surface.

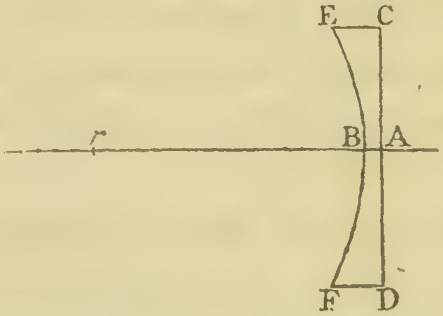
Let *CBD* be a circular arc whose center is *r*, and chord *CD*; draw *rAB* at right angles to *CD*; and the



solid generated by the revolution of the figure *CD*, about the axis *rB*, is called a *plano-convex lens.*

4. A *plano-concave lens* is bounded on one side by a plane, and on the other by a concave spherical surface.

Let *EBF* be a circular arc whose center is *r*; *CAD*

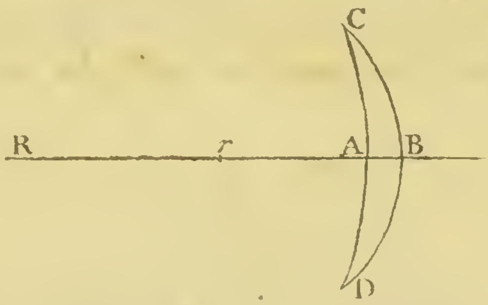


a line line, perpendicular to the radius *rB* produced;

draw *EC*, *FD* parallel to *rA*, and equally distant from it; and the solid generated by the revolution of the figure *ECDF*, about the axis *rA*, is called a *plano-concave lens.*

5. A *meniscus* is bounded by a concave and a convex spherical surface which meet, if continued.

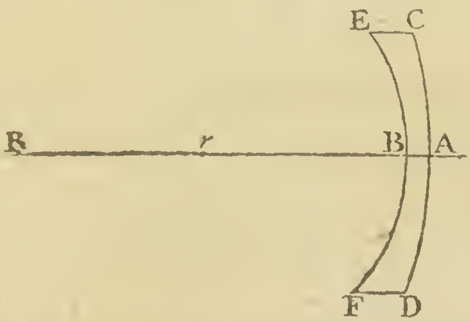
Let *R* and *r* be the centers of two arcs *CAD*, *CBD*, which have their convexities the same way,



and which meet in *C* and *D*; join *Rr* and produce it to *B*; the solid generated by the revolution of the figure *CD*, about the axis *RB*, is called a *meniscus.*

6. A *concavo-convex lens* has also one surface concave and the other convex, but the convex surface, which has the less curvature, does not, if continued, meet the concave surface.

The manner in which the lens is generated is suffi-



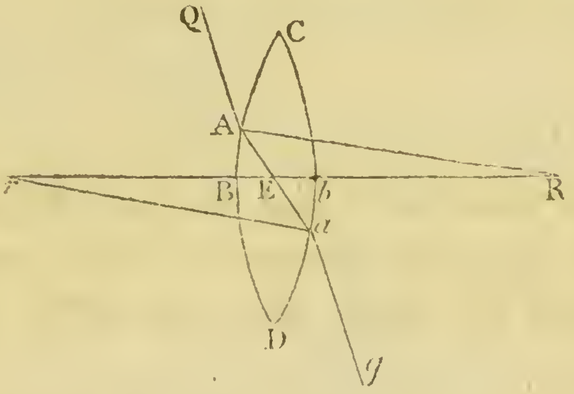
ciently evident from the preceding descriptions.

The thickness of these lenses is, in general, supposed to be inconsiderable.

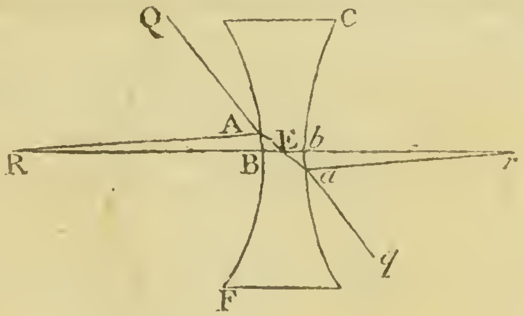
Prop. XL.

(153.) *If the radii* RA, ra, *of the surface of a lens, be drawn parallel to each other, the incident and emergent parts of a ray of light which passes through the lens in the direction* Aa, *will be parallel.*

Join Aa, and suppose two rays Aa, aA to pass out

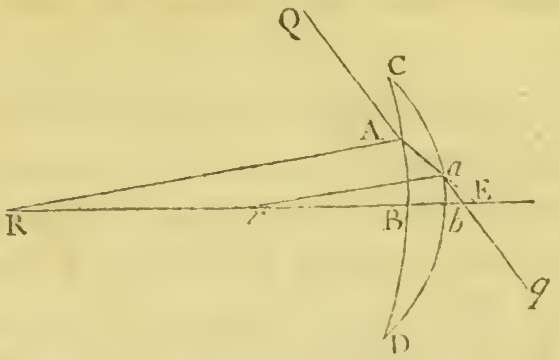


of the lens, in the directions aq, AQ (Art. 29), which are on opposite sides of Aa produced (Art. 33), and in the plane which passes through RA, ra (Art. 24).

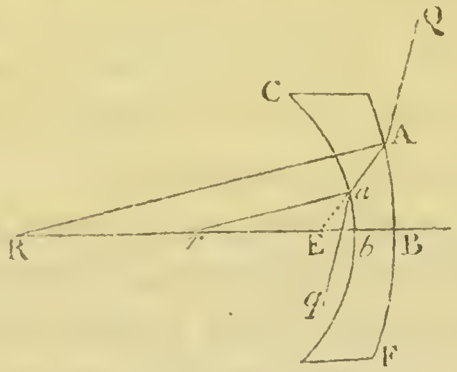


Then, the angles of incidence at A and a being equal,

the angles of deviation are equal; therefore the angles



QAa, Aaq, which are the supplements of the angles



of deviation, are equal; and these are alternate angles; consequently, AQ and aq are parallel.

(154.) DEF. The point E, where Aa, or Aa produced, cuts the axis, is called the *center* of the lens.

(155.) COR. 1. The center E, of the same lens, is a fixed point.

In the similar triangles RAE, raE, $RA : ra :: RE : rE$; therefore, by composition or division, $RA \mp ra : ra :: RE \mp rE\ (Rr) : rE$; the three first terms in which proportion being invariable, the fourth, rE, is also invariable. Thus it appears, that in whatever manner the parallel radii are drawn, Aa, or Aa produced, cuts the axis in the same point.

(156.) COR. 2. In the same triangles, $AE : aE :: RA : ra$; and when A and a coincide with the axis in B and b, $BE : bE :: RB : rb$.

(157.) Cor. 3. The center of the lens is nearer to that surface which has the less radius, or which is the more curved.

For, in the preceding proportion, if rb be less than RB, bE is less than BE.

(158.) Cor. 4. If one radius be increased without limit, the surface to which it belongs becomes plane; and the center of the lens lies in the other surface.

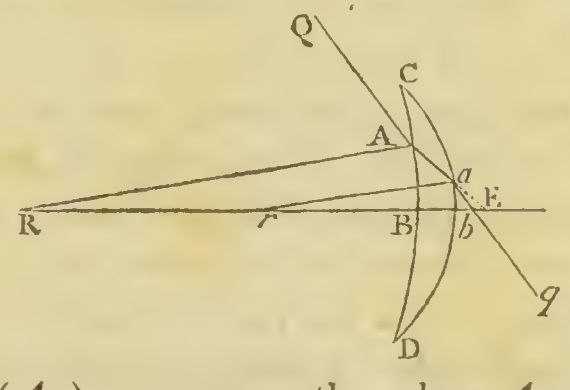
For, if RB be indefinitely greater than rb, BE becomes indefinitely greater than bE, or E coincides with b.

(159.) Cor. 5. The center lies *within* the double convex and double concave lenses, and *without* the meniscus and concavo convex lens.

In the two former cases, the parallel radii RA, ra, lie on opposite sides of the axis; therefore the line which joins the points A and a, cuts the axis. In the two latter cases, the centers R, r, are on the same side of the lens; therefore the parallel radii RA, ra lie on the same side of the axis; consequently Aa must be produced to meet the axis.

(160.) Cor. 6. The center of a meniscus may be at any distance from it's surface.

In the similar triangles RAE, raE, $RA : ra ::$ $AE : aE$; and by division, $RA : RA - ra :: AE :$



$AE - aE$ (Aa); consequently, when A and a coincide with B and b, $RB : RB - rb :: BE : Bb$. If,

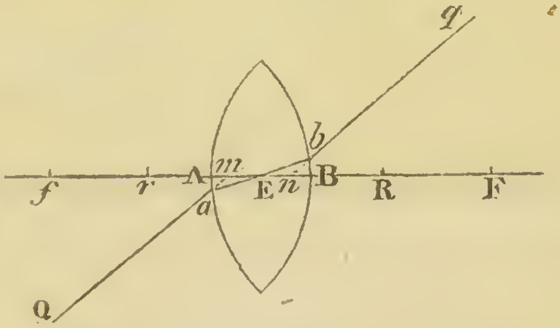
then, the difference of the radii decrease with respect to one of them RB, the distance BE increases with respect to the thickness of the lens; and when RB and rb are equal, BE is indefinitely great.

In the concavo convex lens, when R and r coincide, E coincides with them.

Prop. XLI.

(161.) *If a ray of light* Qabq *be refracted through a lens,* AB, *in the direction* ab *which passes through it's center* E, *to find where the directions of the incident and emergent parts of the ray cut the axis.*

Let Qa, qb, produced if necessary, cut the axis in m and n. Also, let R, r be the centers of the surfaces A, B; F and f the foci of parallel rays incident nearly perpendicularly upon them in the directions fA, FB. Then, $AR : Br :: AE : EB$ (Art. 156.), and consequently, $AR \pm Br : Br :: AB : EB$. Also, Er :



$ER :: Br : AR$ (Art. 155), and $rf : RF :: Br : AR$ (Art. 131); therefore $Er \pm rf\ (Ef) : ER \pm RF\ (EF)$ $:: Er : ER$; hence, $Ef : Ef \pm EF\ (Ff) :: Er : Er \pm ER\ (Rr)$, alt. $Ef : Er :: Ff : Rr$. Now, considering E as the focus of rays incident upon the surface B, $Ef : Er :: EB : En$ (Art. 137); therefore $Ff : Rr :: EB : En$. In the same manner, $Ff : Rr :: EA : Em$.

(162.) Cor. 1. Hence it appears that $EB \stackrel{+}{\sim} EA$ $(AB) : En \stackrel{+}{\sim} Em$ $(mn) :: Ff : Rr$.

(163.) Cor. 2. If AB, the thickness of the lens, be evanescent, $QaEbq$ may be considered as a straight line, unless Ff also vanishes when compared with Rr.

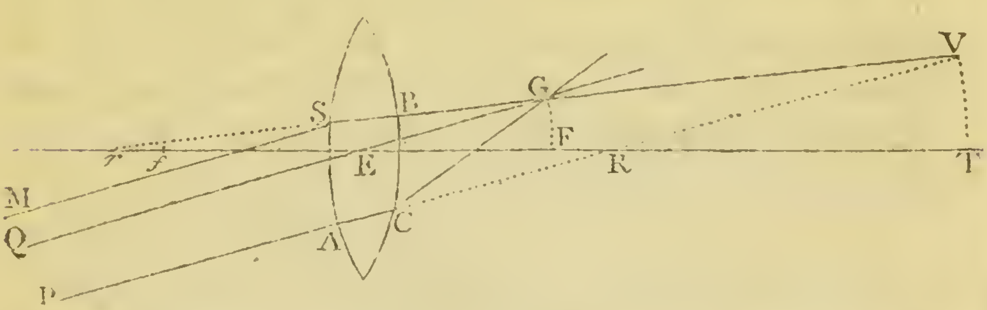
(164.) Cor. 3. If AB be a sphere, E is it's center, and m, and n, coincide with E.

(165.) Def. These points are sometimes called the *focal centers* of the lens.

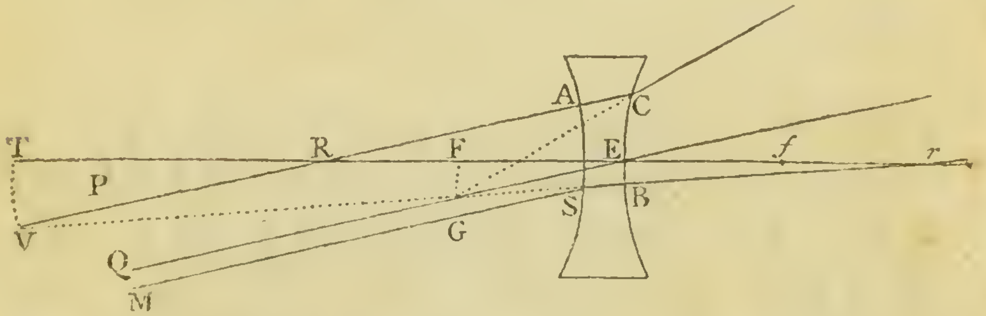
Prop. XLII.

(166.) *To find the principal focus of a lens whose thickness is inconsiderable.*

Let AB be a lens, whose axis is rT, and center E;

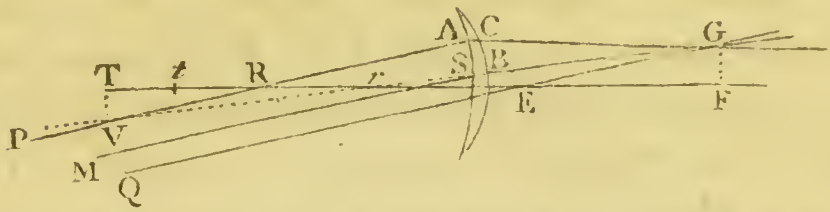


R and r the centers of the surfaces A and B; PA, QE,

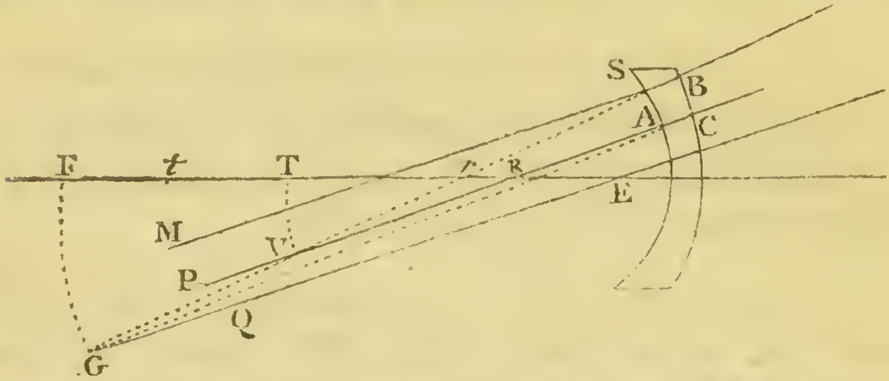


MS a pencil of parallel rays incident upon it; of which QE passes through the center, and may therefore be considered as proceeding in that direction after

the second refraction (Art. 162); consequently, the



focus of emergent rays will be in *QE*, or *QE* produced.



Let *PA* be that ray of the pencil which is incident perpendicularly upon the surface *A*; and in *PA*, or *PA* produced, which passes through *R*, take *AV* : *RV* :: the sine of incidence : the sine of refraction; join *Vr*, and produce it, if necessary, till it cuts *QE* in *G*, and the surface *CB* in *B*. Then, all the rays in the pencil *MSAP*, which are incident nearly perpendicularly upon the surface *AS*, will, after the first refraction, converge to, or diverge from *V* (Art. 130), and in this state they will fall upon the surface *B*; of this pencil, that ray which is incident at *B* coincides with the direction of the radius *rB*, and is therefore incident perpendicularly upon the surface *B*; consequently, it will proceed in the direction *SB* (Art. 27); and the focus of emergent rays will be somewhere in the line *BSV*, or *BSV* produced. The focus will also, as was before observed, be somewhere in *QEG*; therefore *G*, the intersection of the two lines, *QEG* and *BSV*, is the focus of emergent rays.

Now, since RV is parallel to EG, the triangles RVr and EGr are similar; and $Rr : RV :: Er : EG$; alternately, $Rr : Er :: RV : EG$, the distance of the principal focus from the center, or the *focal length* of the lens*.

(167.) Cor. 1. Since $RA \mp rB : rB :: Rr : Er$ (Art. 155), we have also, $RA \mp rB : rB :: RV : EG$.

(168.) Cor. 2. If the inclination of the pencil, to the axis of the lens, be continually diminished, the principal focus G will describe the circular arc GF, whose center is E, and radius EG.

For, since $AV : RV ::$ sin. incidence : sin. refraction $:: m : n$, we have $AR : RV :: m \sim n : n$; and, because AR, m and n are invariable, RV is also invariable. Again, $Rr : Er :: RV : EG$; and, since the three first terms in this proportion are invariable, the fourth, EG, is also invariable.

(169.) Cor. 3. If any point G, in the arc FG, be the focus of rays incident in the contrary direction, these rays will emerge parallel to each other and to GE (Art. 29).

(170.) Cor. 4. It appears from the construction of the figures, that parallel rays are made to converge, by a double convex lens, a plano convex lens, and a

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* It is necessary to observe, that we only determine the ultimate intersection of the rays, when they are incident nearly perpendicularly on each surface; that is, when their inclination to the axis of the lens is diminished without limit. The conclusion is, however, nearly true, when they are inclined at a *small,* though *finite* angle to that axis.

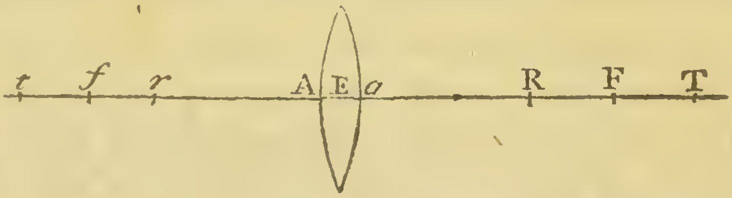
meniscus, of greater density than the surrounding medium. And that they are made to diverge, by a double concave, a plano concave, and a concavo convex lens, of the same description*.

(171.) Cor. 5. If R and r be the radii of the surfaces, $m : n$ the ratio of the sine of incidence to the sine of refraction at the first surface, the focal length of the lens is $\frac{Rr}{R \mp r} \times \frac{n}{m-n}$.

For, $R \mp r : r :: RV : EG$ (Art. 167); and $m-n : n :: R : RV$ (Art. 131); therefore, by compounding these proportions, $\overline{m-n} \times \overline{R \mp r} : nr :: R : EG$; and $EG = \frac{Rr}{R \mp r} \times \frac{n}{m-n}$.

(172.) Cor. 6. The distance of the principal focus from the center is the same on each side of the lens.

Let F be the principal focus when the rays are inci-



dent in the direction tA; f the principal focus when they are incident in the contrary direction.

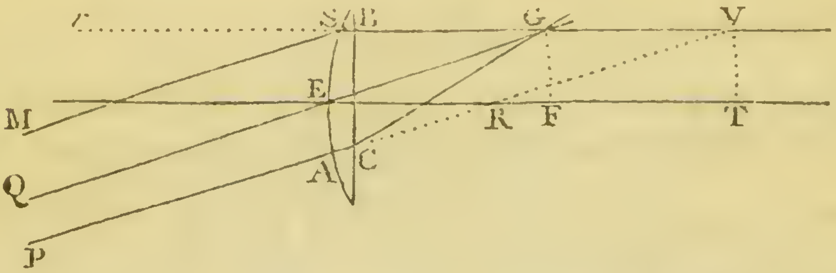
Then, since m and n are the same in both cases, and R and r are alike concerned in the expression $\frac{Rr}{R \mp r}$, the value of $\frac{Rr}{R \mp r} \times \frac{n}{m-n}$ is the same, whether the rays are first incident on the surface A, or on a; that is, $EF = Ef$.

* Lenses are always supposed to be denser than the surrounding medium, unless the contrary be specified.

(173.) Cor. 7. If the radii of the surfaces be given, the focal length varies as $\frac{n}{m-n}$.

(174.) Cor. 8. If one surface be plane, the focal length of the lens is equal to the focal length of the other surface.

When the distance of r is indefinitely increased,



Vr is parallel to ER, and the figure $GERV$ becomes a parallelogram; therefore $EG = RV = R \times \frac{n}{m-n}$.

(175.) Ex. 1. The focal length of a double convex, or double concave glass lens, whose surfaces are equally curved, is equal to the radius of either surface.

In this case, $m : n :: 3 : 2$; and $m - n : n :: 1 : 2 :: AR : RV$ (Fig. Art. 166); hence $RV = 2AR$. Also, $RA + rB : rB :: 2 : 1 :: (RV : EG ::)\ 2AR : EG$; consequently, $EG = AR$.

Ex. 2. If one surface of a glass lens be plane, the focal length is equal to the diameter of the other surface.

Here, $EG = RV$ (Art. 174); that is, $EG = 2AR$.

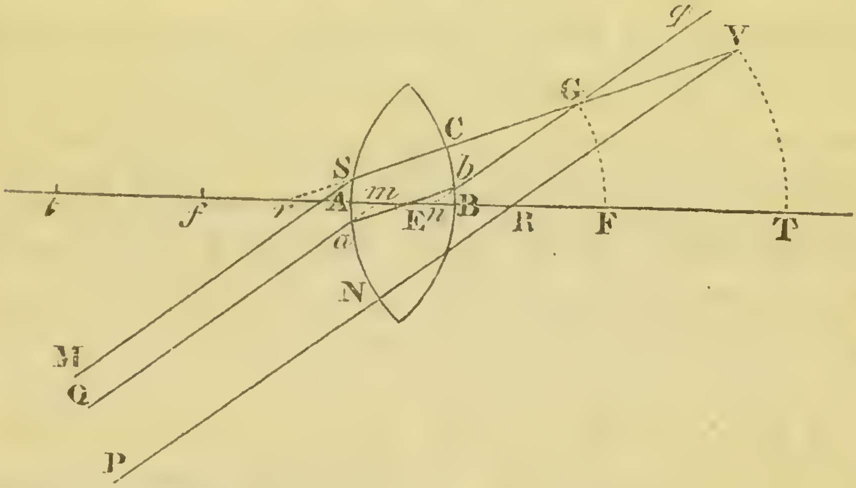
Prop. XLIII.

(176.) *To determine the focal length of a lens whose thickness is not inconsiderable.*

Let AB be the lens whose axis is rR, and center E; R and r the centers of it's surfaces; MS, Qa,

PN a pencil of parallel rays incident upon it, of which $Qabq$ passes through the center; m, n the focal centers; also, let PN be that ray which is incident perpendicularly on the surface A, and therefore PN, or PN produced, passes through R; take $NV : RV ::$ sin. I : sin. R, and V is the focus after the first refraction. Join Vr meeting the surface A in S, and the line bq in G, then G is the focus of emergent rays, as appears from the reasoning in Art. 166.

Then, because nG is parallel to Qa, and therefore to RV, the triangles RrV, nrG are similar, and Rr :



$RV :: nr : nG$, the distance of the principal focus from the focal center n.

(177.) Cor. 1. If the inclination of the pencil to the axis of the lens be diminished continually, the foci V and G will describe the circular arcs VT, GF, whose centers are R and n*; and $Rr : RT :: nr : nF$.

(178.) Cor. 2. If f be the other principal focus, mf is equal to nF.

Since $Rr : RT :: nr : nF$, we have $Rr \times nF = nr \times RT$; in the same manner, when the rays are incident

* See Art. 168.

in the opposite direction, $Rr \times mf = mR \times rt$. Also,

$$En : Em :: rB : RA$$

and, $Er : ER :: rB : RA$; consequently,

$$En \pm Er : Em \pm ER :: rB : RA :: rt : RT^*$$

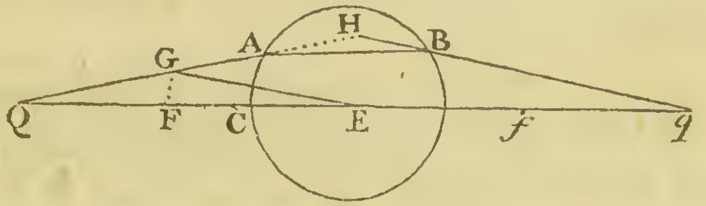
or, $nr : mR :: rt : RT$; hence, $nr \times RT = mR \times rt$; therefore, $Rr \times nF = Rr \times mf$; and $nF = mf$.

(179.) Cor. 3. If any point G, in the arc GF, be the focus of rays incident in the contrary direction, they will emerge parallel to each other, and to Gn†.

Prop. XLIV.

(180.) *When diverging or converging rays are incident nearly perpendicularly upon a sphere, the distances of the focus of incident rays, from the principal focus of rays coming in the contrary direction, from the center of the sphere, and from the geometrical focus of emergent rays, are in continual proportion.*

Let QA and QC be two rays of a pencil incident upon the sphere ACB, of which QE passes through



the center E, and therefore suffers no refraction. Let QA be refracted in the direction AB, and emergent

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* See Art. 131.

† The focal length may also be determined by finding the focus after the first refraction Prop. 32; and this point being the focus of rays incident upon the second surface, the focus of emergent rays may be found by Prop. 28 or 34.

in the direction Bq; produce QA and qB till they meet in H; take F the principal focus of rays incident in the contrary direction; and from the center E, with the radius EF, describe the circular arc FG, cutting QA, or QA produced, in G; join GE. Then, since the ray QA will be refracted in the same manner and degree, whether it be considered as belonging to the focus Q, or to the focus G, Bq is parallel to GE (Art. 150); therefore the triangles QGE, QHq are similar; whence, $QG : QE :: QH : Qq$. Also, since the angles HAB, HBA are equal (Art. 146), $HA = HB$; and when A coincides with C, H coincides with E, and G with F; therefore, ultimately, $QF : QE :: QE : Qq$.

(181.) Cor. 1. From the same similar triangles, $QG : GE :: QH : Hq$; therefore, ultimately, $QF : FE :: QE : Eq$.

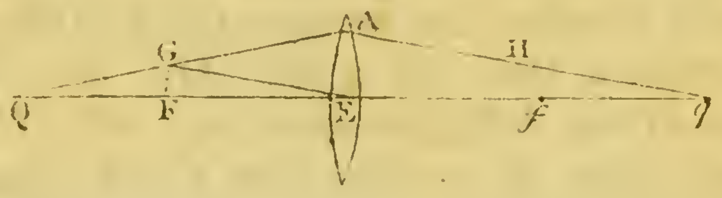
(182.) Cor. 2. If f be the principal focus of rays incident in the direction QC, then, $QF : FE :: Ef : fq$.

For, if q be the focus of incident rays, Q is the focus of refracted rays (Art. 29); therefore $qf : fE :: qE : EQ$ (Art. 181); invertendo, $Ef : fq :: QE : Eq :: QF : FE$; that is, $QF : FE :: Ef : fq$.

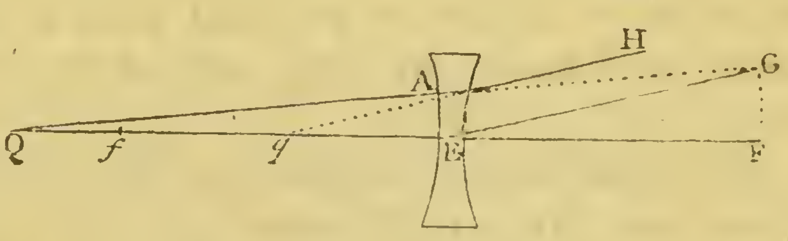
Prop. XLV.

(183.) *When diverging or converging rays are incident nearly perpendicularly upon a lens, whose thickness is inconsiderable, the distances of the focus of incident rays from the principal focus of rays coming in the contrary direction, from the center of the lens, and from the geometrical focus of emergent rays, are in continual proportion.*

Let AE be the lens; E its center; Q the focus of incident rays; QA, QE two rays of the pencil, of which QE is coincident, or nearly coincident with the axis of the lens, and therefore suffers no refraction (Art. 163); let QA be emergent in the direction AH; produce AH backwards or forwards as the case requires, till it meets the axis in q; take F the principal focus of



rays incident the contrary way; and from the center E, with the radius EF, describe the circular arc FG, meeting QA, or QA produced, in G; join GE. Then, since the ray QA will be refracted in the same manner,



and degree, whether it be considered as belonging to the focus Q, or to the focus G, AH is parallel to GE (Art. 169). Hence, the triangles QGE, QAq are similar, and $QG : QE :: QA : Qq$; therefore, ultimately, $QF : QE :: QE : Qq$.

(184.) Cor. 1. In the same triangles, $QG : GE :: QA : Aq$; and ultimately, $QF : FE :: QE : Eq$.

(185.) Cor. 2. Since $QF = QE \pm FE$, we have $QE \pm FE : FE :: QE : Eq$; therefore $Eq = \dfrac{QE \times FE}{QE \pm FE}$;

and $\frac{1}{Eq}=\frac{1}{FE}\pm\frac{1}{QE}$. From this equation, if the nature of the lens be known, any two of the three quantities FE, QE, and Eq being given, the third may be found.

(186.) Cor. 3. If f be the other principal focus, $QF : FE :: Ef : fq$.

For, if q be the focus of incident rays, Q is the focus of refracted rays (Art. 29); therefore $qf : fE :: qE : EQ$ (Art. 184); invertendo, $Ef : fq :: QE : Eq :: QF : FE$; that is, $QF : FE :: Ef : fq$.

Prop. XLVI.

(187.) *The distances* QF *and* Qq *must always be measured in the same direction from* Q.

Since $QF : QE :: QE : Qq$ (Art. 183), we have $QF \times Qq = QE^2$; therefore the sign of the rectangle $QF \times Qq$ is invariable; and, when the distance of Q from F is very great, the distance of q from f is very small (See Art. 186); measuring, therefore, the lines QF and Qq from Q, their rectangle, in this case, is positive, consequently it is always positive, or QF and Qq must always be measured the same way from Q (*Alg.* Art. 471).

(188.) Nearly in the same manner, it may be proved that EQ and Eq must be measured in the same, or opposite directions from E, according as FQ and FE are measured in the same, or opposite directions from F. As also, that QF and fq must be measured in opposite directions from F and f.

(189.) Cor. Because $QF \times fq = FE \times Ef$ (Art. 186), QF varies inversely as fq; and since these distances are measured in opposite directions from F and

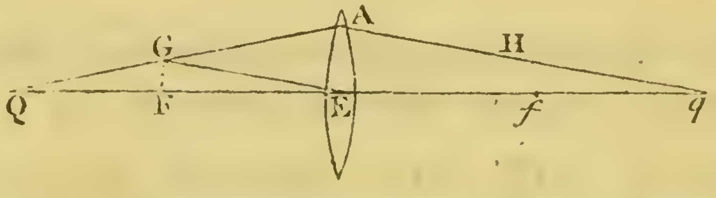
f, it is manifest that the conjugate foci, Q and q, move in the same direction upon the indefinite line QEq.

PROP. XLVII.

(190.) *A convex lens increases the convergency, or diminishes the divergency of rays incident nearly perpendicularly upon it, unless they converge to, or diverge from the center.*

Parallel rays are refracted converging to the principal focus.

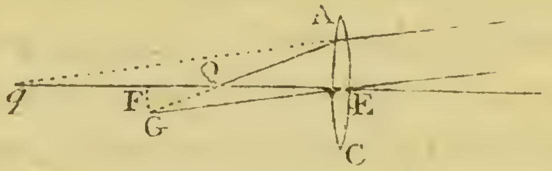
When the incident rays diverge from a point farther



from the lens than it's principal focus, since QF : QE :: QE : Qq, and QF is less than QE, QE is less than Qq; also, QF and Qq are always measured the same way from Q; therefore q is beyond the lens; or the refracted rays converge.

When Q coincides with F, the refracted rays are parallel.

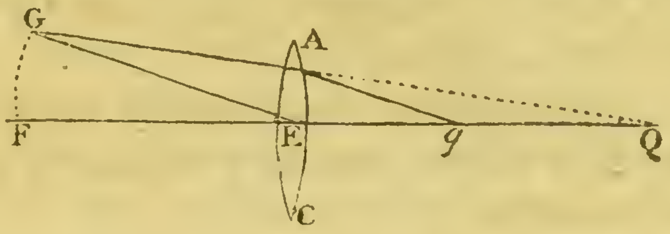
When Q is between F and E, q is on the same side of the lens, and farther from E than Q is (Art. 187);



therefore, the refracted rays diverge less than the incident rays.

When Q coincides with E, q also coincides with it, and the convergency, or divergency, is not altered.

When converging rays are incident upon the lens, QF is greater than QE; therefore QE is greater than



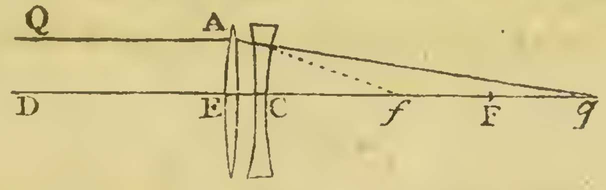
Qq; and q lies between Q and E; consequently, the refracted rays converge more than the incident rays.

(191.) In the same manner it may also be proved, that a concave lens increases the divergency, or diminishes the convergency of rays incident nearly perpendicularly upon it; except when the focus of incident rays coincides with the center of the lens.

Prop. XLVIII.

(192.) *To find the focal length of a compound lens.*

Let the two lenses A and C be placed close together, in such a manner that their axes may coincide; and



let QA and DE be two rays of a parallel pencil incident upon them, of which DE is coincident with their common axis. Take f the principal focus of rays incident upon the lens A, in the direction DE; and F the principal focus of rays incident, the contrary way, upon the lens C. Then, after refraction at the lens A, the rays converge to f, and are thus incident upon the lens C; if, therefore, we take $fF : FC :: Cf : Cq$, and measure Cq and Cf in the same, or opposite

directions from C, according as Ff and FC are measured in the same, or opposite directions from F (Art. 188), q is the focus of emergent rays, and Cq the focal length of the compound lens.

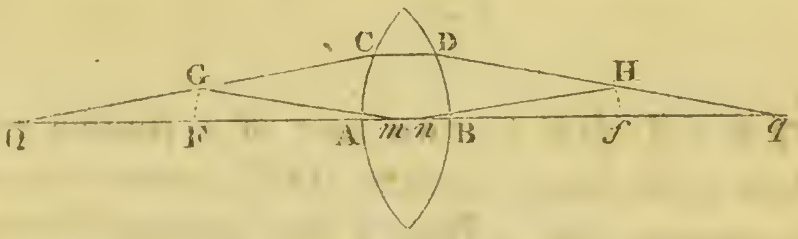
By proceeding in the same manner, we may determine the focal length, when any number of lenses are combined together.

(193.) Cor. When F and f are coincident, the emergent rays are parallel.

Prop. XLIX.

(194.) *When diverging or converging rays are incident upon a lens, whose thickness is not inconsiderable, to find the geometrical focus of emergent rays.*

Let AB be the lens; Qaq it's axis; m, n it's focal centers; F and f, the principal foci of rays incident in the directions qB, QA; QA, QC two rays of the pencil diverging from Q, of which QA is coincident, or nearly coincident with the axis, and therefore suffers no refraction; let QC be refracted in the directions CD, Dq; with the centers m, n, and radii mF, nf,



describe the circular arcs FG, fH meeting QC and Dq in G and H; join Gm, nH.

Then the ray QC will be refracted at C and D, in the same manner and degree, whether we consider it as proceeding from Q or from G, and on the latter supposition Dq is parallel to Gm (Art. 179). Again, if qD be the

incident ray, CQ is the emergent ray, and, as before, CQ is parallel to Hn. Hence it follows, that the triangles QGm, nHq are similar, and $QG : Qm :: nH : nq$; therefore, ultimately, $QF : Qm :: nf : nq$; or, alternately, $QF : nf\ (Fm) :: Qm : nq$.

(195.) Cor. From the same similar triangles, $QG : Gm :: nH : Hq$; ultimately, $QF : Fm :: nF : fq$.

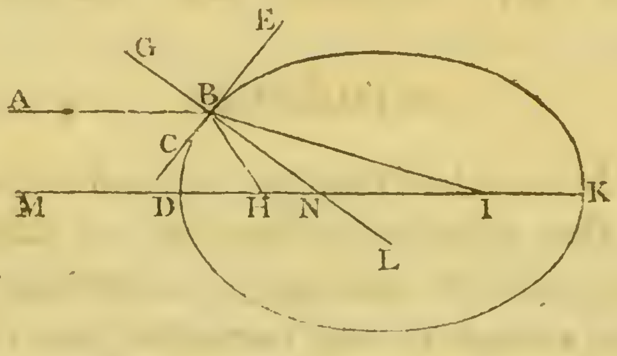
SCHOLIUM.

(196.) It is evident from the 32d and following propositions, that spherical surfaces do not cause all the rays in any pencil to converge to, or diverge from the same point, except in one particular case; and this will be shewn more distinctly in the 7th section.

To remedy the imperfection of optical instruments arising from this cause, it has been proposed to adopt such refractors as are generated by the revolution of the ellipse or hyperbola. But as these are never resorted to in practice, on account of the great difficulty of giving them the exact form, and because the same effect may, in a great measure, be produced by the proper adjustment of the surfaces of spherical refractors, it will be sufficient to explain the geometrical principles upon which the properties of the proposed refractors depend.

(197.) *If a prolate spheroid be generated by an ellipse whose major axis is to the distance between it's foci, as the sine of incidence to the sine of refraction out of the ambient medium into the solid, a pencil of parallel rays, incident in the direction of it's axis, will be refracted, converging accurately, to the farther focus.*

Let *BDK* be the ellipse, by the revolution of which, about it's major axis *DK*, the spheroid is generated; *H* and *I* it's foci; then, by the supposition, *DK* : *HI* :: sin. incidence : sin. refraction. Let *AB*, which is parallel to *DK*, be a ray of light incident upon the spheroid; join *HB*, *IB*; draw *EBC* touching the

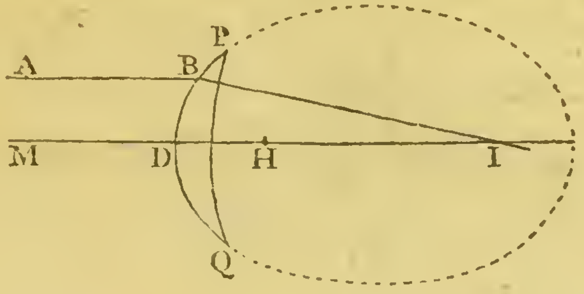


generating ellipse in *B*; through *B* draw *GBL* at right angles to *EBC*, meeting *DK* in *N*.

Then, since the ∠ *HBC* is equal to the ∠ *IBE*, by the nature of the ellipse, and the ∠ *NBC* to the ∠ *NBE*, the angles *HBN*, *NBI* are equal; therefore, *IB* : *BH* :: *IN* : *NH* (Euc. 3. vi.) comp. *IB* : *IB* + *BH* (*DK*) :: *IN* : *IH*, alt. *IB* : *IN* :: *DK* : *IH* :: sin. incidence : sin. refraction; also, *IB* : *IN* :: sin. *INB* : sin. *IBN* :: sin. *BNH*, or sin. *ABG* : sin. *IBL*; therefore, sin. *ABG* : sin. *IBL* :: sin. incidence : sin. refraction; and, since sin. *ABG* is the sine of incidence, sin. *IBL* is the sine of refraction; and because the angle *LBI* is less than a right angle, *BI* is the refracted ray. In the same manner it may be shewn, that every other ray in the pencil will be refracted to *I*.

(198.) Cor. 1. If from the center *I*, with any radius less than *ID*, a circular arc *PQ* be described, the solid

generated by the revolution of *PDQ* about the axis



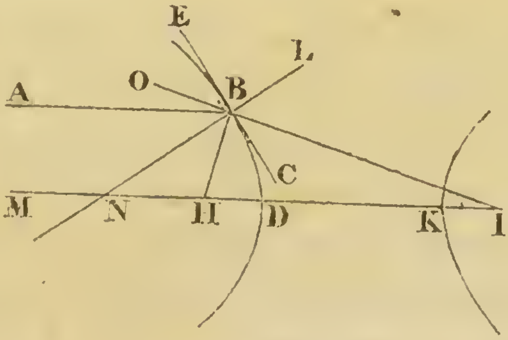
DI, will refract all the rays, incident parallel to *DI*, accurately to *I*.

For, after refraction at the surface *PDQ* the rays converge to *I*; and they suffer no refraction at the surface *PQ*, because they are incident perpendicularly upon it.

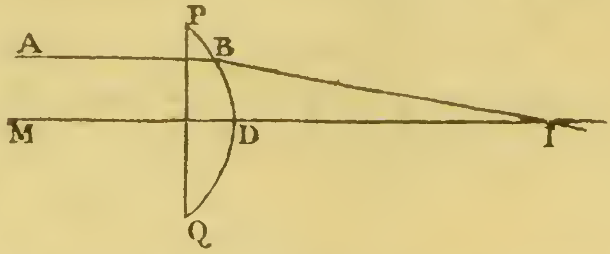
(199.) Cor. 2. Rays diverging from *I* will be refracted parallel to *ID*.

(200.) *If an hyperboloid, whose major axis is to the distance between the foci as the sine of incidence to the sine of refraction out of the solid into the ambient medium, be generated in a similar manner, parallel rays, incident in the direction of the axis, and refracted out of the hyperboloid, will converge to the farther focus.*

The proof is nearly the same as in the former case.



(201.) Cor. 1. If PQ be drawn perpendicular to the axis of the hyperbola, and meet the curve in P and



Q, the solid generated by the revolution of PDQ, about the axis MDI, will refract all the rays, incident parallel to MI, accurately to I.

For, the rays will suffer no refraction at the plane surface PQ.

(202.) Cor. 2. Rays diverging from I, and incident upon the surface PDQ, will be refracted parallel to ID.

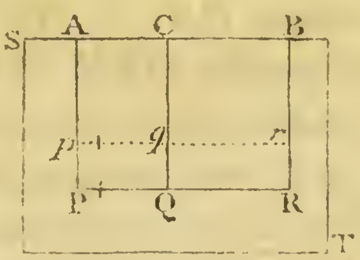
SECT. V.

ON THE IMAGES FORMED BY PLANE AND SPHERICAL REFRACTORS.

PROP. L.

Art. (203.) *THE image of a straight line, formed by a plane refracting surface, is a straight line.*

CASE 1. Let PQR be a straight line, parallel to the plane refracting surface ACB; from P and R, draw



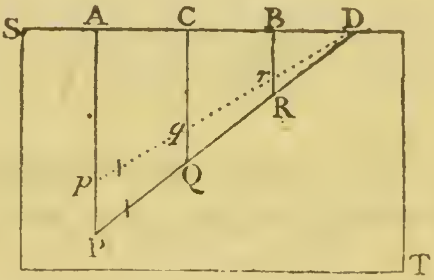
PA, RB, at right angles to AB; and in AP, or AP produced, take $PA : pA ::$ sin. refraction : sin. incidence; and p is the image of P (Art. 120). Draw pr parallel to AB, or PR, and let it meet BR, or BR produced in r; then, since the figures AR, Ar, are parallelograms, $RB = PA$, and $rB = pA$; therefore $RB : rB :: PA : pA ::$ sin. refraction : sin. incidence;

consequently r is the image of R. In the same manner it may be shewn, that the image of any other point Q, is q, the corresponding point in pr, determined by drawing QC perpendicular to AB, and producing it, if necessary, till it meets pr; consequently, pr is the whole image of PR.

In this case, since pR is a parallelogram, the image is equal and parallel to the object.

Case 2. When PQR is inclined to the refracting surface.

Produce PR, if necessary, till it meets the surface in D; from P and R, draw PA, RB, at right angles



to AD; and in AP, or AP produced, take $PA : pA ::$ sin. refraction : sin. incidence; then is p the image of P. Join Dp, cutting BR, or BR produced in r; and in the similar triangles DBR, DAP, $RB : BD :: PA : AD$; also, in the similar triangles DBr, DAp, $BD : rB :: AD : pA$; and, *ex æquo*, $RB : rB :: PA : pA ::$ sin. refraction : sin. incidence; therefore r is the image of R (Art. 120). In the same manner it may be shewn, that the image of any other point Q, in PR, is q, the corresponding point in pr, found by drawing QC perpendicular to AD, and producing CQ if necessary; that is, pr is the whole image of PR.

In this case, $PQ : pq :: QD : qD :: QR : qr$ (Euc. 2. vi.); that is, the corresponding parts of the image and object are proportional.

(204.) Cor. 1. The image and object are on the same side of the refracting surface; and the image is *nearer* to, or *farther* from the surface than the object, according as the rays pass out of a *denser* into a *rarer*, or out of a *rarer* into a *denser* medium.

Ex. If the medium ST be water, contiguous to air, $PA : pA :: 4 : 3$; and $PA : Pp :: 4 : 1$. Thus, the image of the bed of a river is nearer to the surface than the bed itself, by one fourth part of the whole depth.

(205.) Cor. 2. Any two points p, r, in the image, have the same relative situation that the corresponding points P, R, of the object have; therefore the image is erect.

(206.) Cor. 3. If PR, the $\angle PDA$, and the ratio of the sines of incidence and refraction, be known, the $\angle pDA$, and pr may be found.

For, DA being made the radius, tang. PDA : tang. $pDA :: PA : pA ::$ sin. refraction : sin. incidence; therefore the $\angle pDA$ may be found from the tables. Again, $PR : pr :: PD : pD ::$ sec. PDA : sec. pDA.

(207.) Cor. 4. The image of a straight line inclined to the surface, is *greater*, or *less* than the object, according as the rays pass out of a *rarer* into a *denser*, or out of a *denser* into a *rarer* medium.

(208.) Cor. 5. If the figure ST move parallel to itself, on a line which is perpendicular to it's plane, PQR, and pqr, will generate planes, the latter of which is the image of the former.

(209.) Cor. 6. When the object is a plane, parallel to the refracting surface, the image is equal and parallel to the object.

(210.) Cor. 7. If the object be a plane, inclined to the refracting surface, the breadths of the object and image, measured by corresponding lines which are parallel to their common intersection, are equal; but their breadths PR, pr, measured by corresponding lines perpendicular to that intersection, are unequal. In this case, the image and object are not similar.

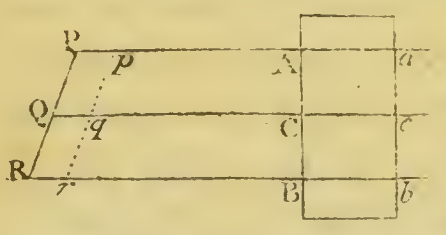
(211.) Cor. 8. If pr be the image of PR, and the eye be so placed as to receive the rays which are incident nearly perpendicularly upon the surface AB, they will enter the eye as if they came from a real object in the situation pr.

Cor. 9. If the rays be refracted at a second surface, pr may be considered as an object placed before that surface, and it's image determined in the same manner.

Prop. LI.

(212.) *The image of a straight line, formed by a medium contained by parallel plane surfaces, is a straight line, equal and parallel to the object.*

Let $ABba$ be the medium, PQR the object placed before it. From P, and R, draw PAa, RBb at right



angles to AB; and in AP, or AP produced, according as Ab is denser, or rarer than the surrounding medium, take $Pp : Aa ::$ sin. incidence $\sim$ sin. refraction : sin. incidence; and p is the image of P (Art. 123). Draw pr parallel to PR, and let it meet BR, or BR pro-

duced, in r. Then, since Pr and Ab are parallelograms, $Rr = Pp$, and $Bb = Aa$; therefore $Rr : Bb :: Pp : Aa ::$ sin. incidence$\sim$sin. refraction : sin. incidence, or r is the image of R. In the same manner it may be shewn, that the image of any other point Q in the object, is q, the corresponding point in pr, determined by drawing QC perpendicular to AB, and producing it, if necessary, till it meets pr. It appears, from the construction, that pr is equal and parallel to PR.

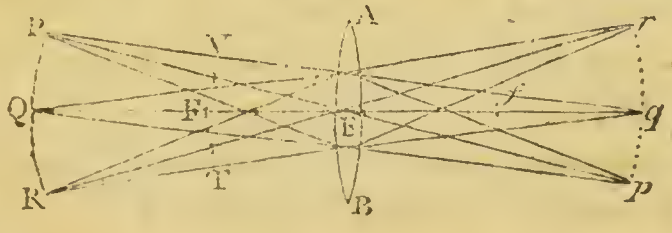
Ex. If the medium Ab be glass, surrounded by air, $Pp : Aa :: 1 : 3$.

(213.) Cor. Whatever be the form of the object, the image will be similar and equal to it (See Art. 71).

Prop. LII.

(214.) *If the object placed before a sphere, or lens whose thickness is inconsiderable, be a circular arc concentric with it, the image will also be a circular arc* concentric with, and similar to the object.*

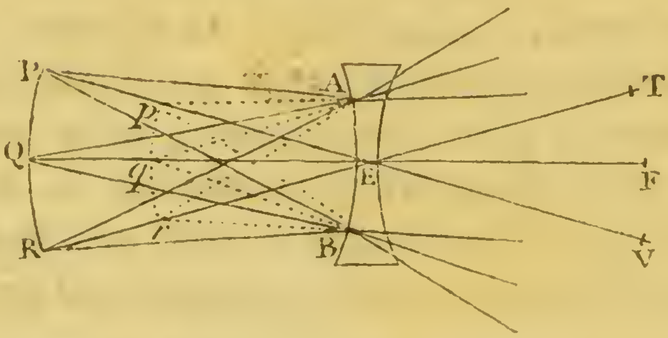
Let AB be the refractor, E it's center; PQR a circular arc whose center is E; in PQR take any



point Q, and join QE; let F be the principal focus of rays incident in the opposite direction to QE. In

* In the case of the lens, the proposition is not accurately true, as appears by the observation contained in the next Note.

QE, or QE produced, take $QF : FE :: QE : Eq$, EQ and Eq being measured in the same, or opposite



directions from E, according as Q and E lie in the same, or opposite directions from F; and q will be the image of Q (Art. 188). From the center E, with the radii EF, Eq, describe the circular arcs VFT, pqr; and from the points P and R, in the object, draw PE, and RE, producing them, if necessary, till they meet pqr in p and r; then will pr be the image of PR. For, since $EP = EQ$, and $EV = EF$, the sum, or difference of EP and EV, is equal to the sum, or difference of QE and EF; that is, $PV = QF$; also, $Ep = Eq$, by the construction; and $QF : FE :: QE : Eq$; therefore $PV : VE :: PE : Ep$, or p is the image of P*. In the same manner it may be shewn, that the image of every other point in PQR, is the corresponding point in pqr; that is, pqr is the whole image of PQR.

The image and object are similar arcs, because they subtend the same, or equal angles at E.

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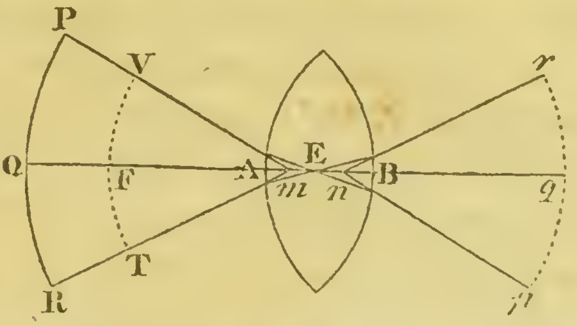
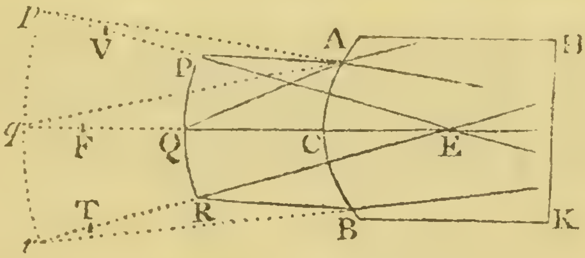
* Here it is supposed that the foci q, p, of direct, and oblique pencils, are equally distant from E. This is not accurately true in the case of the lens, and consequently, the image is not a circular arc; it does not, however, sensibly differ from that form, when the angle which PQR subtends at E is small.

(215.) In the same manner it may be proved, from Art. 194, when the thickness of the lens is not in-



considerable, that if the object be a circular arc whose center is m, the image is a similar arc whose center is n*.

Also, if the refractor be a spherical surface with which the object is concentric, it may be shewn, from



the 139th article, that the image is similar to, and concentric with the object.

(216.) Cor. 1. Since PR and pr are similar arcs, $PR : pr :: EQ : Eq$; hence, in the lens, or sphere, $PR : pr :: QF : FE$ (Arts. 184. 181).

(217.) Cor. 2. If the figure be supposed to revolve about the axis Qq, PQR and pqr will generate similar spherical surfaces, the latter of which is the image of the former.

(218.) Cor. 3. In this case, the magnitude of the object : the magnitude of the image :: $\overline{EQ}^2 : \overline{Eq}^2$.

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* In the subsequent Propositions, the thickness of the lens is supposed to be inconsiderable, unless the contrary be expressed.

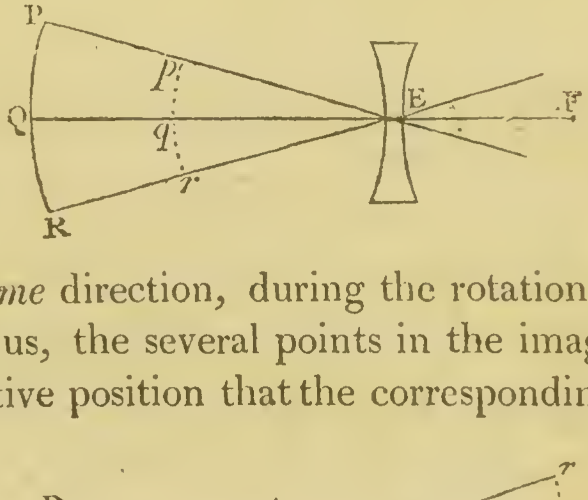
(219.) Cor. 4. If the half of a lens be covered, or cut off by a plane passing through the axis, the image formed by the remaining part will be of the same magnitude, and in the same situation as before; the only alteration produced will be a diminution of the brightness.

The same may be said, if a part of the lens be cut off by any plane which is parallel to the former.

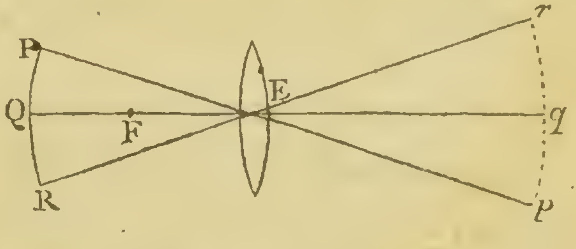
Prop. LIII.

(220.) *When the image and object are on the same side of the center of the refractor, the image is erect with respect to the object; when they are on opposite sides of the center, it is inverted.*

If the line PpE revolve round E, and the point P trace out the object, p will trace out the image. Also, when P and p are on the *same* side of E, they move



in the *same* direction, during the rotation of the line PpE; thus, the several points in the image have the same relative position that the corresponding points in



the object have, or the image is erect. But, when P and p are on *opposite* sides of the center, they move in

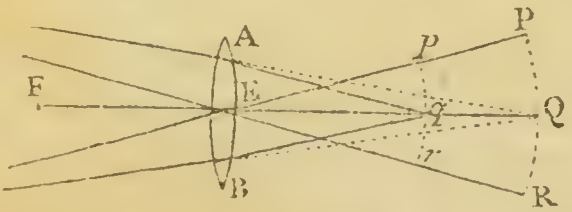
opposite directions, and the situation of any two points in the image is inverted, if compared with the situation of the corresponding points in the object; consequently the whole image is inverted.

(221.) Cor. 1. When the object is placed before a convex lens, and QE is greater than FE, the image is inverted. For, QF and FE, in this case, are measured in opposite directions from F; therefore Q and q are on opposite sides of E (Art. 188). When the object is between F and E, the image is erect.

In the former case, QF may be greater than, equal to, or less than FE; therefore the image may be less than, equal to, or greater than the object (Art. 216). In the latter case, the image is always greater than the object.

(222.) Cor. 2. When the refracted rays actually meet, if a screen be placed at their concourse, an image or picture of the object will be formed upon it. If the screen be placed nearer to the lens, or farther from it, than the focus of refracted rays, the image will be indistinct; because the rays, which proceed from a single point, will be diffused over some space upon the screen, and mixed with the rays which diverge from other points in the object; and this indistinctness will increase as the distance of the screen, from the focus of refracted rays, increases.

(223.) Cor. 3. When converging rays, which tend to form an image PQR, are received by a convex lens



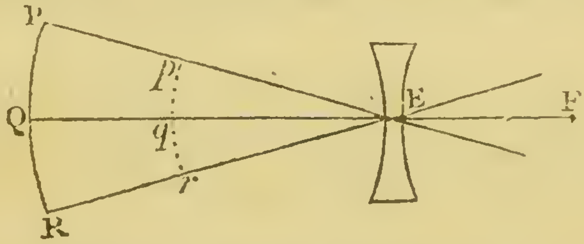
which is concentric with PQR, another image pqr

will be formed, nearer to the lens, and erect with respect to the first image.

Let rays, converging to the several points in PQR, be intercepted by the convex lens AB; take F the principal focus of rays incident in the contrary direction.

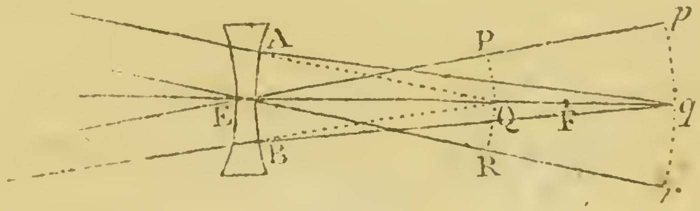
Then, since QF is greater than FE, QE is greater than Eq; consequently, PQR is greater than pqr. Also, since Q and E are on the same side of F, Q and q are on the same side of the center E; and therefore, pqr is erect with respect to PQR.

(224.) Cor. 4. When the object is placed before a double concave lens, since QF and FE are measured



the same way from F, Q and q are on the same side of E; that is, the image is erect. Also, since QF is greater than FE, the object is greater than the image.

(225.) Cor. 5. If converging rays, which tend to form the image PQR, be intercepted by a double



concave lens concentric with PQR, the second image pqr will be *erect*, or *inverted*, with respect to the first, according as Q is *nearer to*, or *farther from*, the lens than F, the principal focus of rays incident in the contrary direction.

When Q is between F and E, EQ and Eq are measured in the same direction from E (Art. 188); consequently, pqr is erect with respect to PQR. When F is between Q and E, PQR and pqr are on opposite sides of the center; and therefore the image pqr is inverted.

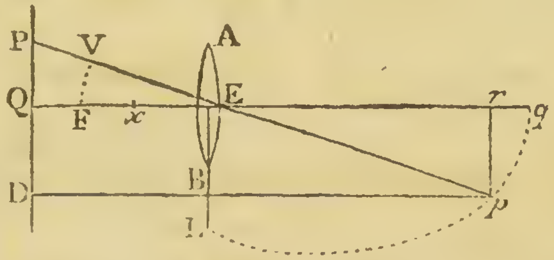
(226.) Cor. 6. When Q lies between F and E, or pqr is erect, QF is less than FE, and consequently EQ is less than Eq, or PQR is less than pqr. In other cases, PQR may be greater than, equal to, or less than pqr.

(227.) Cor. 7. When Q coincides with F, the emergent rays, in each pencil, are parallel.

Prop. LIV.

(228.) *The image of a straight line, formed by a lens or sphere, is the arc of a conic section.*

Let AB be a lens, or sphere, whose center is E; PD a straight line placed before it; through E, draw QEq at right angles to PQ; in PD take any point



P; join PE, and produce it. Let F be the principal focus of rays incident in the direction qE; with the center E, and radius EF, describe the circular arc FV, cutting PE in V. In PEp, take $PV : PE :: PE : Pp$, measuring PV and Pp in the same direc-

tion from P; then p is the image of P (Art. 187 *). Draw pD parallel to qQ. Then, since the triangles PEQ, PpD are similar, $PE : Pp :: QE : Dp$; consequently, $PV : PE :: QE : Dp$. Also, $PV : VE :: PE : Ep$ (Arts. 184. 181); alternately, $PV : PE :: VE\ (FE) : Ep$; therefore $QE : Dp :: FE : Ep$; and alternately, $QE : FE :: Dp : Ep$; consequently the locus of the point p, is a conic section, whose focus is E, and directrix PD †.

(229.) Cor. 1. The curve is an *ellipse*, *parabola*, or *hyperbola*, according as QE is *greater* than, *equal* to, or *less* than FE.

(230.) Cor. 2. When Ep coincides with EL, that ordinate to the axis which passes through the focus, Dp becomes equal to QE, and therefore $EL = EF$; that is, half the latus rectum of the conic section is equal to the focal length of the glass.

(231.) Cor. 3. The curvature of the image, at it's vertex, is the same, wherever the object is placed.

(232.) Cor. 4. If xq be the major axis of the conic section, $Qq : Eq :: QE : FE$; and by division, or composition, $QE : Eq :: QF : FE$; therefore $Eq = \dfrac{QE \times FE}{QF} = \dfrac{QE \times FE}{QE \mp FE}$. In the same manner, $Ex = \dfrac{QE \times FE}{QE \mp FE}$; consequently, $xq = \dfrac{QE \times FE}{QE \sim FE} \pm \dfrac{QE \times FE}{QE + FE} = \dfrac{2\,QE^2 \times FE}{QE^2 \sim FE^2}$. Also, $xE \times Eq$, the square of the semi-axis minor, $= \dfrac{QE^2 \times FE^2}{QE^2 \sim FE^2}$.

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* See Note, p. 93. † See Art. 93.

(233.) Cor. 5. If the focal length of the refractor be finite, and pr be drawn perpendicular to the axis, the evanescent arc pq is equal to pr; and $QP : qp :: EQ : Eq$.

Also, whilst the angle QEP, which QP subtends at the center of the glass, is small, though finite, the image pq, when formed at a finite distance from the refractor, will, as to sense, be a right line, and $QP : qp :: EQ : Eq$.

Prop. LV.

(234.) *The sun's image, formed by a spherical refracting surface, lens or sphere, is a circle, and nearly in the principal focus of the refractor.*

Let E be the center of the refractor; F and f it's principal foci; PQ a radius of the sun's disc. Then,

since FE is inconsiderable with respect to QF, the image of Q, may, for all practical purposes, be considered as coincident with the principal focus f of the refractor (Arts. 140. 182. 186); also, since QP subtends a small angle at E*, it's image, fp, may be considered as a straight line (Art. 233). Now, let the figure revolve about Qf as an axis, and whilst QP generates the circle which represents the sun's disc, fp will generate it's image, which is, therefore, a circle.

* About 16′.

In the same manner it may be shewn, that the sun's image, formed by a spherical reflector, is a circle, and in the principal focus of the reflector.

(235.) Cor. 1. Since the angle fEp is given, fp the radius of the image, is proportional to Ef, the focal length of the glass.

(236.) Cor. 2. The area of the image varies as the square of it's radius; and therefore as the square of the focal length of the reflector, or refractor.

(237.) Def. By the reflecting, or refracting *powers* of different substances, we understand the ratio of the number of rays reflected, or transmitted by them, if the number of incident rays be the same.

Thus, if one surface reflect two thirds, and another one third of the incident rays, the reflecting *powers* are said to be as 2 : 1.

(238.) Cor. The number of rays reflected, or transmitted, varies as the number incident, and the reflecting, or refracting power, jointly.

For, if the number of incident rays be given, the number reflected, or transmitted, varies as the power; if the power be the same, the number of rays reflected, or transmitted, varies as the number incident; therefore, when both vary, the number of rays reflected, or transmitted, varies as the number incident, and the reflecting, or refracting power, jointly.

Prop. LIV.

(239.) *The density of rays in the sun's image varies directly as the area of the aperture of the reflector, or refractor by which it is formed, and the reflecting, or refracting power, jointly; and inversely as the square of the focal length of the reflector, or refractor.*

The density of rays in the image varies directly as their number, and inversely as the space over which they are diffused*; that is, directly as the number, and inversely as the square of the focal length of the reflector, or refractor (Art. 236). Also, the number of rays reflected, or transmitted, varies as the number incident, and the reflecting, or refracting power, jointly; that is, as the area of the aperture through which the incident rays pass, and the power, jointly; consequently, the density of rays in the image, varies, directly, in the compound ratio of the aperture and power, and inversely, as the square of the focal length of the reflector, or refractor†.

(240.) Cor. 1. When the apertures are circular, the density varies, directly, in the compound ratio of the square of the linear aperture and power; and inversely as the square of the focal length of the reflector, or refractor.

(241.) Cor. 2. If the radii of the surfaces of a concave reflector, and a double convex lens of glass, be equal, as well as their apertures and powers, since the focal length of the reflector : the focal length of the lens :: 1 : 2 (Arts. 45. 175), the density of rays in the image formed by the reflector : the density in the image formed by the lens :: 4 : 1.

(242.) Cor. 3. The focal length of a glass sphere, is three times as great as the focal length of a reflector of the same radius (Art. 148); therefore, on the

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* Here we suppose the rays to be *uniformly* diffused over the image, which is not the case; it is, however, true at points similarly situated in images formed by rays which are diffused according to the same law.

† The rays lost in passing through the air, are not taken into the account.

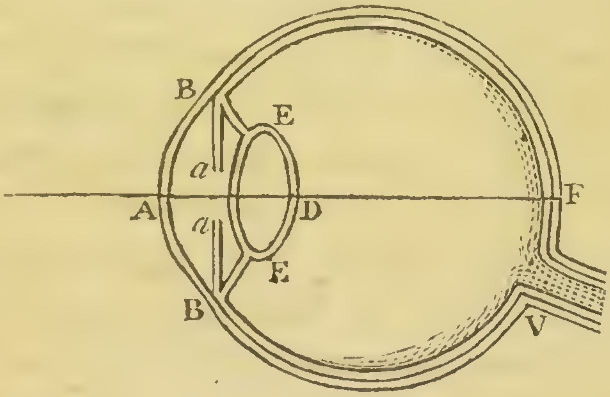
former supposition, the density of rays in the image formed by the reflector : the density in the image formed by the sphere :: 9 : 1.

(243.) Cor. 4. If the rays which tend to form the sun's image, be received by a double convex lens, another image, nearer to the lens, and consequently less than the former, will be produced (Art. 223). Hence it appears, that independent of the rays lost in their passage through the lens thus employed, the burning power of a reflector, or refractor may be increased.

Sect. VI.

ON THE EYE AND THEORY OF VISION.

Art. (244.) The annexed figure represents a section of the human eye, made by a plane which is perpendicular to the surfaces of the coats which contain it's several humours, and also to the nose. It's form is



nearly spherical, and would be exactly so, were not the forepart a little more convex than the remainder; the parts *BFB*, *BAB*, are, in reality, segments of a greater and a less sphere.

The humours of the eye are contained in a firm coat *BFBA*, called the *sclerotica*; the more convex, or protuberant part of which, *BAB*, is transparent, and from it's consistency, and horny appearance, it is

called the *cornea*. This coat is represented by the space contained between the two exterior circles *BFBA*.

Contiguous to the sclerotica is a second coat of a softer substance, called the *choroeides*. This coat is represented by the next white space, and extends, along the back part of the sclerotica, to the cornea.

From the junction of the choroeides and cornea arises the *uvea*, *Ba*, *Ba*, a flat, opaque membrane, in the forepart of which, and nearly in it's center *, is a circular aperture called the *pupil*.

The pupil is capable of being enlarged, or contracted with great readiness †; by which means, a greater or less number of rays may be admitted into the eye, as the circumstances of vision require. In weak light, too few rays might render objects indistinct; and in strong light, too many might injure the organ. Whilst the pupil is thus enlarged, or contracted, it's figure remains unaltered. This remarkable effect is thought to be produced by means of small fibres which arise from the outer circumference of the uvea, and tend towards it's center; this circumference is also *supposed* to be muscular, and by it's equal action upon the fibres, on each side, the form of the pupil is preserved, whilst it's diameter is enlarged, or contracted.

At the back part of the eye, a little nearer to the nose than the point which is opposite to the pupil, enters the *optic nerve V*, which spreads itself over the whole of the choroeides like a fine net; and from this

* In some eyes, the pupil is a little nearer to the nose than the center of the uvea.

† The limit of it's aperture, in the eyes of adult persons, appears to be from about $\frac{1}{5}$ to $\frac{1}{10}$ or $\frac{1}{12}$ of an inch. Harris's *Optics*, p. 94.

circumstance is called the *retina*. It is immersed in a dark mucus which adheres to the choroeides.

These three coats, the *sclerotica*, the *choroeides*, and the *retina*, enter the socket of the eye at the same place. The sclerotica is a continuation of the *dura mater*, a thick membrane which lies immediately under the scull. The choroeides is a continuation of the *pia mater*, a fine thin membrane which adheres closely to the brain. The retina proceeds from the brain.

Within the eye, a little behind the pupil, is a soft transparent substance *EDE*, nearly of the form of a double convex lens, the anterior surface of which is less curved than the posterior, and rounded off at the edges, *E*, *E*, as the figure represents. This humour, which is nearly of the consistency of hard jelly, decreasing gradually in density from the center to the circumference, is called the *crystalline* humour. It is kept in it's place by a muscle, called the *ligamentum ciliare*, which takes it's rise from the junction of the choroeides and cornea, and is a little convex towards the uvea*.

The cavity of the eye, between the cornea and the crystalline humour, is filled with a transparent fluid like water, called the *aqueous humour*. The cavity between the crystalline humour and the back part of the eye, is also filled with a transparent fluid, rather

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* The anterior surface of this muscle, and the posterior surface of the uvea, are covered with a black mucus, evidently designed to absorb any of the extreme rays which may happen to reach so far, and which might be reflected to the retina, and produce confusion in the vision.

more viscous than the former, called the *vitreous humour*.

(245.) It is not easy to ascertain, with great accuracy, the refracting powers of the several humours; the refracting powers of the aqueous and vitreous humours, are nearly equal to that of water; the refracting power of the crystalline humour is somewhat greater*.

(246.) The surfaces of the several humours of the eye are so situated as to have one line perpendicular to them all. This line *ADF* is called the axis of the eye, or the *optic axis* †.

(247.) The point in the axis at which the object, and the image upon the retina, subtend equal angles, is not far distant from the posterior surface of the crystalline lens ‡, though it's situation is subject to a small change, as the figure of the eye, or the distance of the object is changed.

(248.) From the consideration of the structure of the eye, we may easily understand how notices of external objects are conveyed to the brain.

Let *PQR* be an object, towards which the axis of the eye is directed; then, the rays which diverge from any point *Q*, and fall upon the convex surface of the aqueous humour §, have a degree of convergency given

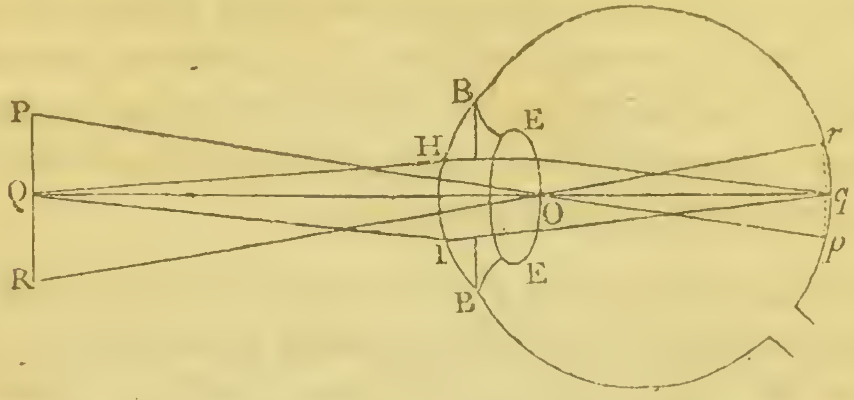
* This is manifest from the figure of the crystalline humour, and the circumstance that persons couched, (in which case the crystalline lens is taken out) are obliged to use convex glasses.

† The dimensions of the several parts of the eye may be seen in Harris's *Optics*, page 94.

‡ Harris, p. 97.

§ The surfaces of the cornea are nearly parallel to each other, and therefore it produces little alteration in the divergency of rays which pass through it.

them; they are then refracted by a double convex lens, denser than the ambient mediums, which in-



creases the convergency; and if the extreme rays *QH*, *QI*, have a proper degree of divergency before incidence, the pencil will be again collected upon the retina, at *q*, and there form an image of *Q*. In the same manner, the rays which diverge from any other points, *P*, *R*, in the object, will be collected at the corresponding points *p*, *r*, of the retina, and a complete image, *pqr*, of the object *PQR*, will be formed there. The impression, thus made, is conveyed to the brain by the optic nerve, which originates there, and is evidently calculated to answer this purpose.

(249.) Since the axes of the several pencils cross each other within the eye, (see Art. 247), the image upon the retina is inverted with respect to the object*; and if, by any means, the image of an erect object, be erect upon the retina, that object appears inverted.

(250.) It has been objected, that if the images upon the retina be inverted, external objects ought to appear inverted. To which it may be answered, that experience

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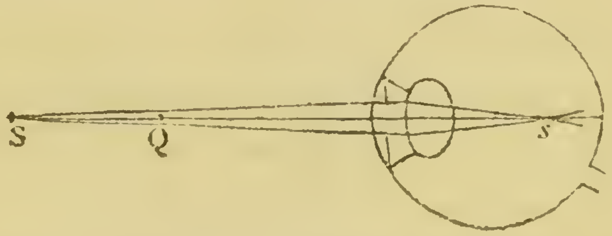
* If the outer coat be taken from an ox's eye, whilst it is warm, the images of external objects are observed to be inverted upon the retina.

alone teaches us, what situation of the external object corresponds to a particular impression upon the retina; nor is it of any consequence what that impression is, or in what manner it is made; but whenever the same effect is produced upon the organ, we expect to find the same external object, and in the situation to which our former experience directs us.

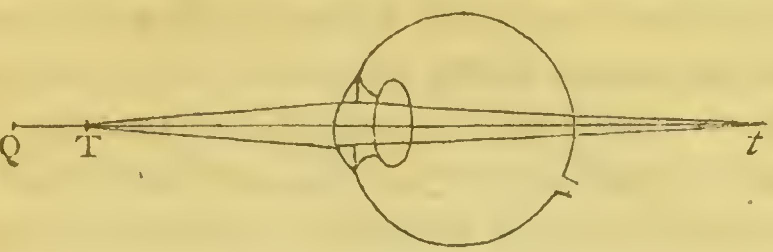
(251.) If the point *P* move along the line *PQR*, from the right to the left, the image *p* moves from the left to the right, upon the retina. And in general, whenever the image, upon the retina, moves from the left to the right, we are led, by experience, to conclude that the object really moves from the right to the left.

(252.) If the form of the eye, the situation of the several humours, and their respective surfaces, remain unaltered, it is manifest that those rays only, which diverge from points at a particular distance, can be collected upon the retina. Thus, if the image of *Q* be formed exactly upon the retina, the image of *S*, a



point farther from the eye than *Q*, will be formed within the eye; therefore, the rays which proceed from this point, will be diffused over some space upon the retina; and, if they are mixed with the rays which diverge from other points in the object, necessary to be distinguished from the former, the vision will be

indistinct*. The rays which diverge from T, a point nearer to the eye than Q, will, after refraction, con-



verge to t, a point behind the retina; in this case also, they will be diffused over some space upon the retina, and the vision, as before, will be indistinct.

(253.) By what change in the conformation of the eye, we are enabled to see objects distinctly at different distances, is not fully ascertained. The fact itself is sufficiently manifest; but authors differ in opinion as to the manner in which the effect is produced. It is supposed by some, that the general figure of the eye is altered; that, when the object to be viewed is near, the length of the eye, measured along the axis, is increased by the lateral pressure of external muscles; and, on the contrary, when the object is remote, that the length of the eye is diminished, by the relaxation of that pressure. Others suppose the effect to be produced by a change in the place, or figure of the crystalline humour. Others, by an alteration in the diameter of the pupil. Others ascribe the effect to a change in the curvature of the cornea.

Much stress cannot be laid upon the first of these causes, as distinguished from the last, since it's existence

* In many cases it is not necessary to distinguish very nicely the adjacent parts of objects; as in reading large print, viewing trees, houses, mountains, &c. and though the rays are not exactly collected upon the retina, the image is sufficiently well defined for the purpose.

is not proved by experiment; and there is no necessity for recurring to a bare hypothesis of this kind. With respect to the second, the *ligamentum ciliare* does not appear sufficiently strong to produce any considerable change in the form, or situation of the crystalline humour. And as it is clearly ascertained*, that persons couched can see distinctly at different distances, we must conclude that the effect is not to be ascribed to any change in this humour.

A change in the aperture of the pupil has very little effect in rendering objects distinct at different distances.

It has been seen before (Art. 219), that if the position of an object, placed before a lens, be given, the image is formed at the same distance, whether the rays proceeding from the object pass through a greater or smaller portion of the lens. The same is manifestly true of the eye, so long as the several mediums of which it consists, retain their forms and positions; the expansion, therefore, or contraction of the iris cannot cause the images, of objects at different distances from the eye, to be formed on the retina. If, however, lateral rays are stopped by the iris, the indistinctness arising from the diffusion of the rays in each pencil, over some space on the retina, will be lessened; but this can only take place in a small degree, and when the objects are very near to the eye†.

* *Philos. Trans.* Vol. lxxxv. p. 6.

† It is on this account that a small hole in a thin plate enables us to view objects at a less distance than we could with the naked eye, as it answers the purpose of a farther contraction of the pupil, and excludes those rays in each pencil, which diverge too much. This

assistance

The principal change by which the effect is produced, seems to be an alteration in the curvature of the cornea. In order to shew that such a change takes place, Mr. Ramsden fixed the head of a spectator so securely, that no deception could arise from it's motion, and directed him to look at a distant object; whilst the eye was in this situation, he placed a microscope, in such a manner, that the wire, with which it was furnished, apparently coincided with the outer surface of the cornea; and then directing the spectator to look at a nearer object, he found that the cornea immediately projected beyond the wire of the microscope*.

Now, when the distance of an object is diminished, supposing no alteration to take place in the eye, the divergency of the extreme rays of the pencil incident upon the pupil, is increased; and therefore, if the image of the object in the first situation, be formed upon the retina, in the latter it will be formed behind it (Art. 252); but an increase in the curvature of the cornea will increase the convergency of the refracted rays, or bring them sooner to a focus; and thus, by a proper change in this coat of the eye, the rays will again be brought to a focus upon the retina, and the object be still seen distinctly.

(254.) The least distance at which objects can be seen distinctly by common eyes, is about 7 or 8

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assistance cannot be made use of to any great extent, because the image upon the retina will soon become indistinct for want of light; and the inflection of rays at the sides of the hole, will render it confused.

* This experiment is described by Mr. Home, in a very ingenious paper on the subject, *Philosoph. Trans.* Vol. lxxxv. p. 16.

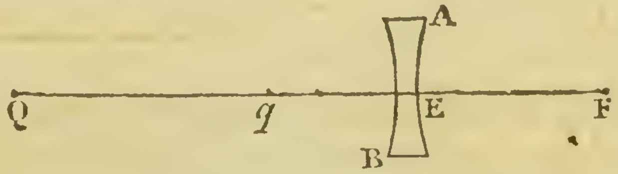
inches*. The greatest distance cannot be so easily, or accurately ascertained. It seems that the generality of eyes are capable of collecting parallel rays upon the retina, or so near to it as to produce distinct vision; and thus, the greatest distance at which objects can be distinctly viewed, is unlimited. For this reason, in adapting optical instruments to common eyes, and calculating their powers, we suppose the parts to be so arranged, that the rays in *each pencil* may, when they fall upon the cornea, be parallel.

(255.) If the humours of the eye be too convex, parallel rays, and such pencils as diverge from points at any considerable distance, are collected before they reach the retina (Art. 252); and objects, to be seen distinctly, must be brought nearer to the eye. This inconvenience may be remedied by a concave glass whose focal length is so adjusted as to give the rays, proceeding from a distant object, such a degree of divergency as the eye requires.

Prop. LV.

(256.) *Having given the distance at which a short sighted person can see distinctly, to find the focal length of a glass which will enable him to see distinctly at any other given distance.*

If qE be the distance at which he can see distinctly,



and QE a greater distance, at which he wishes to view

* *Harris*, p. 124.

objects; let AB be a concave lens, whose focal length is such, that the rays which are incident upon it, diverging from Q, may, after refraction, diverge from q; then they will have a proper degree of divergency for the eye of this spectator. Take F the principal focus of rays incident in the contrary direction; then, since Q and q are conjugate foci, $QF : QE :: QE : Qq$ (Art. 183); *dividendo*, $FE : QE :: Eq : Qq$; therefore $FE = \dfrac{QE \times Eq}{Qq}$.

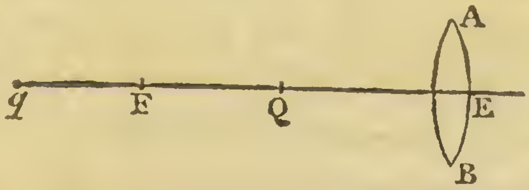
(257.) Cor. If QE be indefinitely great, $FE = Eq$.

(258.) When the humours of the eye are too flat, the rays which diverge from a point near the eye, converge to a point behind the retina. This imperfection may be remedied by a convex lens, whose focal length is adjusted to the distance at which objects are to be viewed, and the degree of convergency in the rays of each pencil, which the eye requires.

Prop. LVI.

(259.) *Having given the distance at which a long sighted person can see distinctly, to find the focal length of a glass which will enable him to see distinctly at any other given distance.*

If qE be the distance at which he can see distinctly, and QE the distance at which he wishes to view



objects, let AB be a convex lens whose focal length FE, is such, that the rays which diverge from Q, may,

after refraction, diverge from q. Take F the principal focus of rays incident in the contrary direction; and since Q and q are conjugate foci, $QF : QE :: QE : Qq$; *componendo*, $FE : QE :: Eq : Qq$; and $FE = \frac{QE \times Eq}{Qq}$.

(260.) Cor. 1. If qE be indefinitely great, or the eye require parallel rays, $FE = QE$.

(261.) Cor. 2. If the eye require converging rays, q falls on the other side of the lens; in this case, FE is less than QE.

(262.) In the choice of glasses for long, or short sighted persons, care should be taken to select such as have the least refracting power that will answer the purpose. For, the eye has a tendency to retain that conformation to which it is accustomed; and therefore, by the use of improper glasses, it's imperfection may be increased.

Prop. LVII.

(263.) *If the apparent distance of an object be given, and the angle which it subtends at the center of the eye be small, it's apparent linear magnitude is nearly proportional to that angle.*

When objects are at the same distance from the eye, and appear to be so, we learn by experience to form an estimate of their linear magnitudes with considerable accuracy. That is, the apparent magnitudes are nearly proportional to the real magnitudes, and the *real* magnitudes are proportional to the angles which the objects subtend at the center of the eye, when those angles are small; therefore their *apparent* magnitudes are nearly in that ratio.

(264.) An object, and it's image upon the retina, subtend equal angles at the center of the eye (Art. 247); and supposing the center fixed, and the angles small, the linear magnitude of the image is nearly proportional to the angle which it subtends at that center; therefore the linear magnitude of an object at a given distance from the eye, is nearly proportional to the linear magnitude of it's picture upon the retina*.

(265.) The judgment we form of the magnitude of an object, depends very much upon the notion we have of it's distance; and since the apparent distance depends upon a variety of causes, which are subject to no calculation, in speaking of apparent magnitude authors generally suppose the apparent distance to be given.

(266.) Def. By the *visual angle* of an object, we understand the angle which the axes of the extreme pencils coming from it, contain at the center of the eye; whether the object is viewed with the naked eye, or with the assistance of reflecting surfaces, or refracting mediums.

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* On this account, perhaps, we learn to estimate the magnitudes of objects, at a given distance, more readily than we should otherwise be able to do; but, did the magnitude of the picture upon the retina vary according to any other law, we should still learn by experience to estimate magnitudes by the sight; that is, the apparent and real magnitudes would still be proportional.

When objects subtend considerable angles at the center of the eye, we judge of their magnitudes by carrying the optic axes over their several parts; and in this case also, the apparent, and real magnitudes are nearly proportional, if we have had sufficient experience in estimating magnitudes of this description.

Prop. LVIII.

(267.) *When a given object is viewed with the naked eye, the density of light in the image upon the retina, supposing none to be lost in it's passage through the air, and the diameter of the pupil to be invariāble, is nearly the same at all distances of the eye from the object.*

The density of light, in the image of a small portion of the object, varies directly as the number of rays, and inversely as the space over which they are diffused. The number of rays which pass through the pupil, supposing it's diameter given, and that none are stopped in their progress, varies inversely as the square of the distance of the object from the eye (Art. 11). Also, the linear magnitude of the picture upon the retina, varies as the angle which the object subtends at the center of the eye, nearly (Art. 264); that is, nearly in the inverse ratio of the distance of the object from the eye; consequently, the area of the picture upon the retina, or space over which the rays are diffused, varies inversely as the square of that distance, nearly. Hence it follows, that the density of rays in the image, varies inversely as the square of the distance of the object from the eye, on one account, and directly as the square of that distance on the other; therefore, upon the whole, the density is invariable.

What has been proved of the image of one small portion of the object, may be proved of every other; consequently, the density, in every part of the image upon the retina, is invariable.

SCHOLIUM.

(268.) It may here be observed, that a considerable quantity of light is lost, or absorbed, in it's passage through the air; and that the quantity thus lost, *cæteris paribus*, increases as the distance between the object and the eye increases, though not in that ratio*. On this account, therefore, the brightness of an object decreases, as it's distance from the eye increases. As the distance of the object increases, however, the aperture of the pupil is enlarged; and therefore more rays are, by this means, received into the eye; and thus the former effect is, in some degree counteracted.

Did the density of rays in the picture upon the retina decrease considerably, as the distance of the object increases, bodies in the neighbourhood of the spectator would, by their superior brightness, overpower the impressions made by those which are more

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* If the spaces, through which the light passes, increase in arithmetical progression, the quantity of light will decrease in geometrical progression.

Let the space be divided into equal portions; and let A, B, C, D, &c. represent the quantity of light which enters the 1st, 2d, 3d, 4th, &c. portion, respectively; also, suppose $\frac{1}{m}$th part of the whole light to be lost, or absorbed in it's passage through the 1st portion of space; then $\frac{1}{m}$th part of the remainder will be lost in passing through the 2d; and so on. Thus $A-\frac{A}{m}$, or $\frac{m-1}{m}\times A=B$. In the same manner, $\frac{m-1}{m}\times B=C$; $\frac{m-1}{m}\times C=D$, &c. That is, A, B, C, D, &c. form a decreasing geometrical progression, whose common ratio is $\frac{m-1}{m}$.

remote; and the latter would be discerned with great difficulty, or not at all. We are indeed able to distinguish objects in exceedingly different degrees of light, at different times; thus we are able to read a small print by moon light, though it's intensity does not exceed $\frac{1}{90,000}$th part of the intensity of common day-light*. But this quantity of light is not sufficient to render such objects discernible as are surrounded by others much more luminous; for, the strong light proceeding from the latter bodies, by the powerful impression it makes upon the retina, overcomes the effect produced by the more delicate pencils which flow from the former, as weaker sounds are not distinguishable in a hurricane. This seems to be the reason that the flame of a candle is scarcely discernible in broad day-light; and that stars become visible at different times after sun-set, according to their different degrees of brightness.

(269.) The impressions made by rays of light upon the retina continue some time after the impulses cease, as appears by the experiment of a burning coal, whirled round in a circle, which was mentioned on a former occasion (Art. 8). Sir Isaac Newton accounts for this phenomenon by supposing that the impressions of light are conveyed to the brain by vibrations excited in the retina, and propagated, through the optic nerve, to the sensorium †; and that the vibrations once produced, continue some time, perhaps about 1″, after the exciting cause has ceased to act.

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* This is Dr. Smith's *calculation.* M. Bouguer concludes from *experiment,* that the strength of moon-light is about $\frac{1}{300,000}$th part of that of day-light.

† *Optics,* Query 16.

(270.) In explaining the nature and circumstances of vision, we have only to attend to the structure of one eye; for, in whatever manner rays are refracted, and images formed by the humours of one eye, in the same manner will the same effects be produced by the humours of the other. The only question that can arise is, how it happens that in vision with both eyes, objects appear single. It is not easy to decide, whether this effect is produced by the similarity of corresponding parts of the optic nerves, and their union in the brain*, or by habit. In support of the latter opinion, we may be allowed to alledge the following fact, related by Mr. Chesselden: A person had one of his eyes distorted by a blow; and, for some time, every object, to him, appeared double; but by degrees the most familiar ones became single, and in time, all objects became so, though the distortion continued †. To this we may add, that children sometimes *learn* to squint; and by proper attention, this habit may again, in a great measure, be corrected. Under both circumstances, objects appear single, and it is manifest that the images cannot, in both cases, fall upon corresponding points of the retinas.

(271.) But, whatever be the cause of single vision, when an object is viewed attentively, the axes of both eyes are, in general ‡, directed to it. Thus, if P be the object, the eyes are moved till the optic axes, AP,

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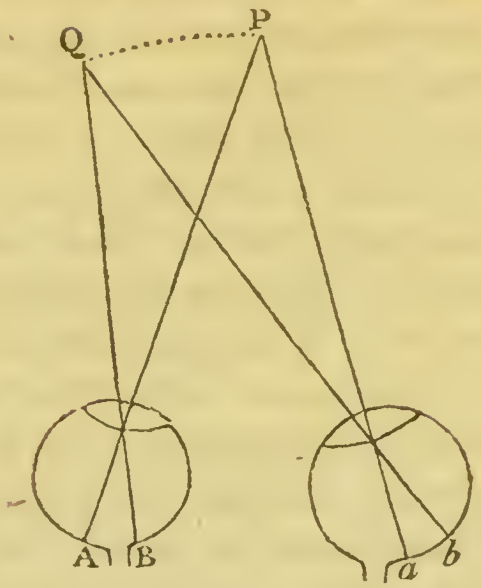
* *Optics,* Query 15.

† Smith's *Optics,* Art. 137.

‡ Persons who squint do not direct the optic axes to the object they are looking at.

aP, meet in *P*; and the images, *A*, *a*, are formed on

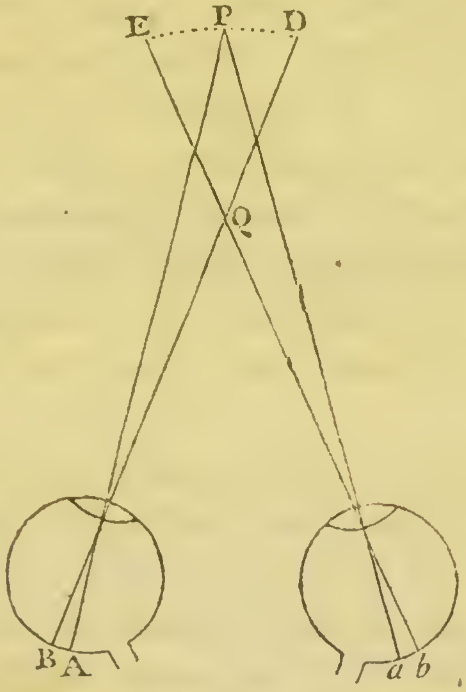


corresponding points of the retinas*. In this position of the images, whether from the correspondence of the nerves, or from experience, the idea of a single object is suggested to the mind; scarcely differing from the idea excited by one of the images alone, excepting that the object appears somewhat brighter when seen with both eyes, than when seen with one. Also, whilst the eyes remain in the same position, the images, *B*, *b*, of *Q*, an object near to *P*, and at the same distance from the eyes, will be formed on the retinas; the eyes having assumed a proper conformation for distinct vision at that distance; and *B*, *b*, which are

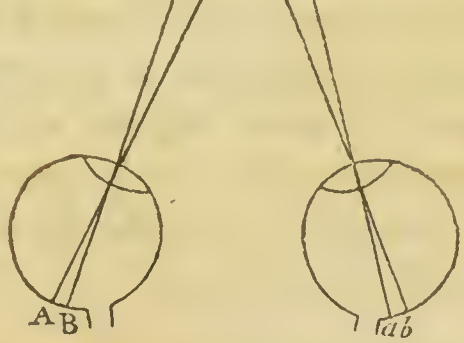
* The retina may perhaps be most susceptible of the impressions of light where the optic axis meets it; and the images formed near that part of the retina, will be less distorted and more regularly and distinctly defined, than when the rays pass more obliquely through the humours of the eye. For one or both of these reasons, we direct the axis of each eye to an object, when we wish to view it to the greatest advantage.

both on the right, or both on the left of the respective axes, are corresponding points, and suggest the idea of a single object, as in the former case.

But, if an object *Q* lie between the optic axes, or



those axes produced, it's images will be formed at ***B***, ***b***, on points upon the retinas which do not correspond;



and thus they will excite the sensations, usually produced by different objects, *D*, *E*, at that distance to

which the eyes are adapted for vision. However, when the attention is more particularly called to the object Q, the optic axes are directed to it, and the points, D, E, coincide.

(272.) Of apparent distance, measured in a direct line from the spectator, nothing has been said in the foregoing section. It is subject to no calculation, and therefore, does not immediately fall in with the plan of the present work. It may not however be improper to enumerate the causes upon which it depends, as they are given by Mr. Harris in his Optics, referring the reader, for farther information on the subject, to that work*.

1. *The change of conformation in the eye, is one of the means whereby we are enabled to judge of small distances.*

In viewing near objects, or such as are within about an arm's length of us, at every sensible change of distance, the eye must also change it's conformation for procuring distinct vision. And thus, if we were accustomed to look attentively at objects with one eye only, it is very likely that the changes made on these occasions, would be sensible enough, after repeated trials, to enable us to judge pretty accurately of the different degrees of small distances.

This method, however, will not serve us to a greater extent than, perhaps, about 20 or 30 inches. Beyond which limit, the different degrees of divergency of rays in the different pencils which enter the eye, bear no sensible proportion to the different distances of the points from which they diverge.

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* Page 154. . . . 168

2. *Inclination of the optic axes, is another more certain mean of distinguishing degrees of small distances.*

When an object is viewed attentively with both eyes, the axis of each is directed towards it; and as the distance of the object is increased or diminished, there is a corresponding diminution or increase in the angle at which these axes are inclined to each other. The sensations which accompany these different inclinations, enable us to determine with considerable accuracy, the places of objects which are not above five or six feet from us. As the distance becomes greater, we begin to be more uncertain in our estimations of it; and beyond three or four yards, the means, hitherto considered, seem to be of little or no use. For, beyond that extent, the differences of the optic angles, arising from the different distances, are so very small as to become in a manner insensible.

3. *The length of the ground plane, or the number of intervening parts perceived in it, is another mean, by which we estimate distance.*

When the floor, or ground on which we stand, is uniformly extended before us, in a line produced directly from us on this plane, we can distinguish, that such successive parts as form sensible angles at the eye, are successive, or one behind the other; and the greater the number of visible parts which the line contains, the greater, consequently, is the visible extent of the whole.

Again, a row of houses, columns, or trees regularly planted, appears longer than a plain wall of the same extent. For, the more visible and remarkable parts in the former case, enable us to correct the estimate we make when such objects do not intervene; and

also, our previous knowledge that the several intermediate objects are disposed at equal intervals, tends to protract the apparent length of the whole chain still farther. A river, at first, looks not so broad, as after we have had a side view of the bridge across it: and indeed, a given extent of water, does not appear so long as the same extent of land; as it is more difficult to distinguish parts in the surface of the one, than it is in the surface of the other.

4. *Different degrees of apparent distance are suggested by the different appearances of known objects, or by the known magnitudes of their least visible parts.*

A building, none of whose parts are discernible, appears much farther off than another, whose windows and doors are visible; and this latter appears farther off than a house having visible parts which are known to be still smaller; as the bricks in the wall, tiles on the roof, &c. Objects of unusual magnitudes, detached as it were, from others, mislead us in our judgement of distances; the greater magnitude usually suggesting to the mind, the idea of less distance.

5. *All other things being the same, different colours and degrees of brightness of objects, cause a difference of apparent distance.*

As objects become more and more remote, the light, which arrives at the eye from their least visible parts, is continually diminished (Art. 268); and they appear more faint, languid, and obscure. Also, their colours not only gradually lose their lustre, but likewise degenerate from their native hue, and participate more of the blueish colour of the sky, as the rays have passed through a greater body of air, or as the images upon

the retina are tinged with a greater proportion of sky light.

These different appearances are of use to us in judging of the real distances of known objects; and consequently affect the ideas of apparent distances; those objects that are brightest, and whose colours are most vivid, appearing nearest. Thus in foggy weather all objects appear farther off than ordinary; the diminution of light, in this case, producing the effect of that diminution which arises from greater distance.

(273.) When objects, which subtend small angles at the center of the eye, are of the same colour and brightness, and at the same distance, their apparent magnitudes are proportional to those angles (Art. 263). And when they are at different distances, and subtend equal angles at the center of the eye, since their *real* magnitudes are proportional to their *real* distances, it is probable that their *apparent* magnitudes are nearly proportional to their *apparent* distances*. And thus, in general, the apparent magnitudes are as the visual angles, and apparent distances, jointly.

Hence it follows, that any error in our estimate of apparent distance, will produce a proportional error in our estimate of magnitude. Thus, in foggy weather, at the same time that objects appear farther off, they appear larger; and the diameter of the sun, or moon, appears greater, or less, according as we are led by circumstances to suppose it's distance greater or less at one time than at another.

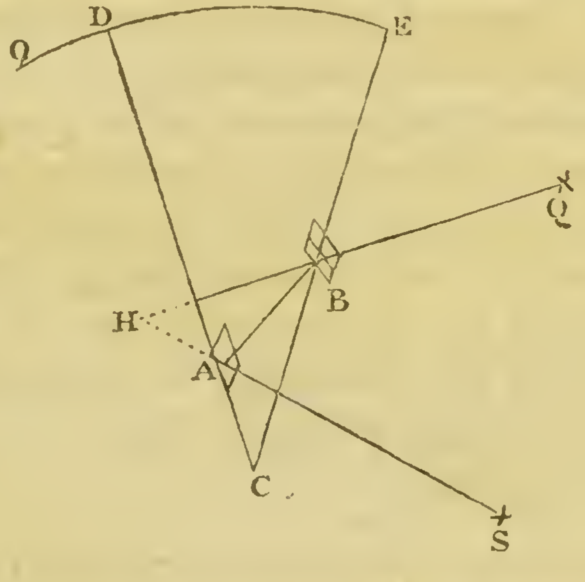
* Any error in this proposition must arise from the limited experience we have of the truth of the former. The magnitude of which we are here speaking, is the *linear* magnitude.

SECT. VII.

ON OPTICAL INSTRUMENTS.

On HADLEY'S *Quadrant.*

Art. (274.) UPON the radii *DC*, *EC*, of the quadrant *OEC*, and at right angles to it's plane, are fixed two plane reflectors *A*, *B*, whose surfaces are



parallel when the index *D*, on the moveable radius *CD*, is brought to *O*; and consequently, the arc *OD* will measure their inclination when the moveable

radius *CD* is in any other situation. The whole surface of the glass *B* is not quicksilvered, a part of it being left transparent that objects may be seen directly through it, and by rays which pass close to the quicksilvered, or reflecting part.

When the angular distance of two objects, *S*, *Q*, is to be taken, the quadrant is held in such a position that it's plane passes through them both; and the radius *CD* is moved till one of them *S* is seen, after two reflections of the incident ray *SA*, in the direction *HQ*; and the other *Q*, by the direct ray *QH*, in the same line; that is, till the objects apparently coincide. Then, if *SA* be produced till it meets *QH* in *H*, the angle *SHQ*, contained between the first incident, and last reflected ray, is equal to twice the angle of inclination of the two reflectors (Art. 39); therefore, the angular distance of the two objects is measured by twice the arc *DO* *.

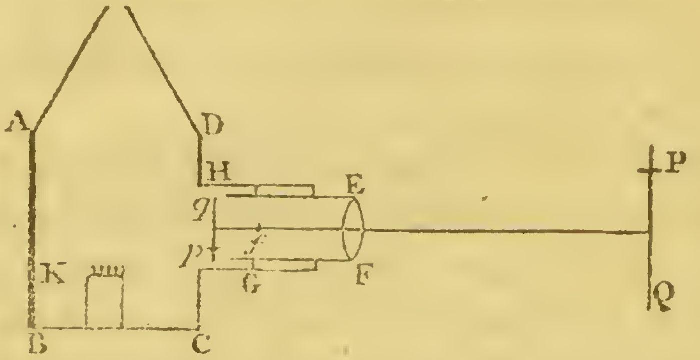
On the Magic Lantern.

(275.) The figure *ABCD* represents a tin box, or lantern, in the fore part of which is a sliding tube, furnished with a double convex lens *EF*. Between the lantern and the lens, a small space, *qp*, is left to admit a thin plate of glass, upon which inverted figures are painted in transparent colours.

When this instrument is used, the lamp *K* being lighted, and the room darkened, the tube is moved, till *qp* is farther from the lens than it's principal focus

* The method of adjusting this instrument may be seen in Mr. Vince's *Practical Astronomy*, p. 8.

f; and consequently PQ, an inverted image of pq, or an erect image of the figure intended to be represented,



is formed at some distance from the lens (Art. 221), and painted in it's proper colours upon a screen placed at the concourse of the refracted rays.

Sometimes a reflector is placed behind the lamp, or a convex lens before it, for the purpose of throwing a greater quantity of light upon pq.

(276.) Cor. If the screen and lantern be fixed, and their distance exceed four times the focal length of the lens, the image may be thrown upon the screen, by moving the lens nearer to, or farther from pq, as the case requires.

For, $qf : qE :: qE : qQ$ (Art. 183), or $qE - fE : qE :: qE : qQ$; in which proportion there is only one unknown quantity, qE, which may be determined by the solution of a quadratic equation whose roots are possible, except the distance qQ be less than four times the focal length of the lens EF*.

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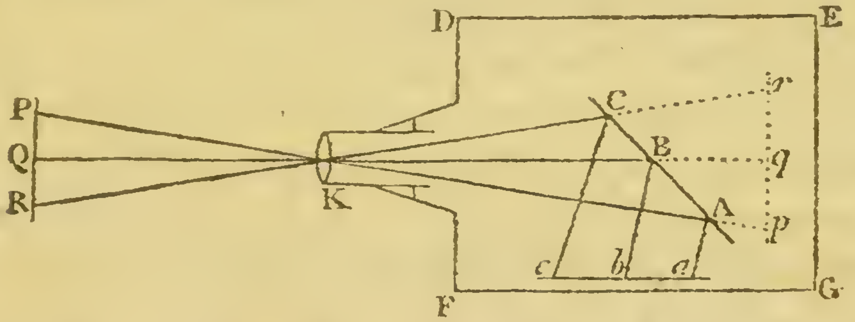
* Let $qE = x$; $fE = a$; $qQ = b$. Then, $x - a : x :: x : b$; therefore $x^2 = bx - ba$; or, $x^2 - bx = -ba$; hence, $x^2 - bx + \frac{b^2}{4} = \frac{b^2 - 4ba}{4}$; and $x = \frac{b}{2} \pm \frac{\sqrt{b^2 - 4ba}}{2}$. If $b = 4a$, we have $x = 2a$. If b be less than $4a$, the expression is impossible, which shews that the image cannot, in this case, be formed upon the screen.

On the Camera Obscura.

(277.) If light be admitted, through a convex lens, into a darkened chamber, or into a box from which all extraneous light is excluded, and the refracted rays be received upon a screen, placed at a proper distance, inverted images of external objects will be formed upon it. And if the lens be fixed in a sliding tube, the images of objects at different distances may successively be thrown upon the screen, by moving the lens backwards or forwards, as in the magic lantern.

Let *PQR* be an object at a considerable distance from the lens, and at right angles to it's axis; the image *pqr*, will be formed, nearly in the principal

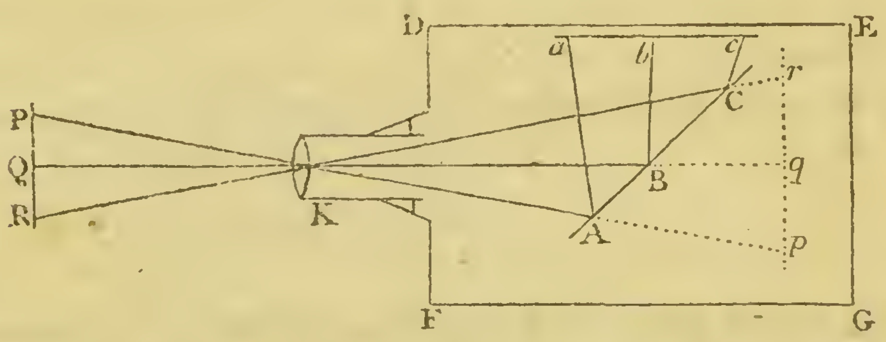


focus of the glass, and may be received upon a screen placed there. But, in general, the rays are intercepted before they form the image, by a plane reflector, *AC*, inclined at an angle of 45° to the axis of the lens; or, which is the same thing, at an angle of 45° to the image *pqr*; by this means, an image *abc* is formed, similar and equal to *pqr*, and inclined at an equal angle to the reflector (Art. 72), and consequently, parallel to the axis of the lens. If the lens be moved towards the reflector, the image *pqr*, (which is in the

principal focus of the lens,) and consequently, *abc*, will move from the reflector; and the contrary. Thus, the image *abc*, may, by a proper adjustment of the lens, be thrown upon a table, or any surface prepared to receive it.

(278.) Cor. 1. When the image is thrown downwards, it will appear erect to a spectator between *DF* and the reflector *ABC*; since the point *a*, which corresponds to *p*, or to *P* in the object, is nearest to the reflector. Also, the axis of a pencil of rays which flows from a point to the right of *Q*, will cross the axis *Qq* at the center of the lens, and the rays will form an image which, to this spectator, is to the right of *b*.

(279.) Cor. 2. When the image is thrown upwards, if it be viewed by a spectator whose face is turned towards the object, it will appear erect as to the



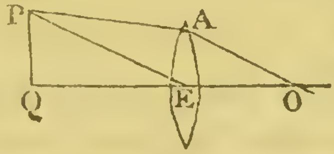
top and the bottom; for the point *a*, which corresponds to *p*, or to *P* in the object, is the farthest point in the image from the reflector; and therefore, the farthest point from the spectator. But, with respect to the right and the left, the image will be inverted. For, the axis of a pencil of rays which flows from a point in the object to the right of *Q*, will cross the axis *Qq* at the center of the lens, and the rays will be

collected at a point which, to this spectator, is to the left of *b*.

Prop. LIX.

(280.) *An object may be seen distinctly through a convex lens.*

Let *AE* be a convex lens* ; *E* it's center ; *PQ* an object placed in it's principal focus. Then, the rays



which diverge from any point *P* will be refracted parallel to each other, and to *PE* (Art. 169) ; and therefore, they will be proper for vision to common eyes. In the same manner, the rays diverging from any other point will be refracted parallel to each other, and the whole object will be seen distinctly (Art. 254).

If the eye require diverging rays, the object must be placed between the lens and it's principal focus ; for then, the rays which diverge from *P*, a point between the principal focus and the glass, will, after refraction diverge (Art. 190) ; and therefore be proper for distinct vision in this case.

If the eye require converging rays, the object must be placed beyond the principal focus ; for then, the rays in each pencil will, after refraction, converge.

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* Of this description are the double convex lens, the plano convex, and the meniscus.

Prop. LX.

(281.) *When an object is placed in the principal focus of a convex lens through which it is viewed, it appears erect.*

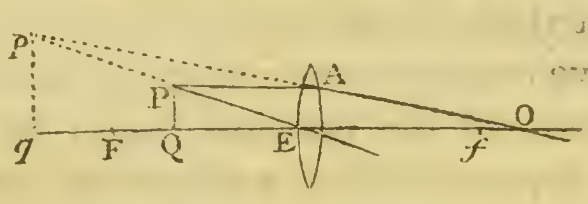
For, if AO be drawn parallel to PE (Fig. Art. 280), the rays which diverge from P, and are received by the eye, enter the pupil in the direction AO; and the rays which diverge from Q, enter the pupil in the direction EO. Thus, the rays which flow from the extreme points P, Q, of the object, cross each other at O, and therefore the picture upon the retina is inverted; or, the object appears erect (Art. 249).

(282.) Cor. In the same manner, if the object be *near* to the principal focus, and the eye not very distant from the glass, the image upon the retina will be inverted; and consequently, the object will appear erect.

Prop. LXI.

(283.) *To determine the variation of the angle which a given object subtends at the center of the eye, when viewed through a convex lens.*

Let E be the center of the lens; F and f it's principal foci; PQ the given object; pq it's image; O



the place of the eye; join Op, and let it meet the lens in A. Then, the rays which diverge from P, enter the eye in the direction AO; and those which diverge

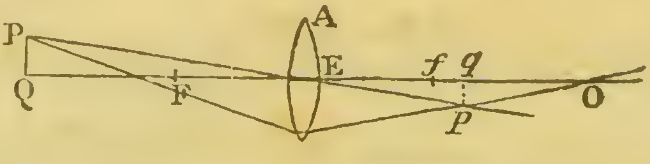
from Q, enter the eye in the direction EO; therefore, the $\angle AOE$, or pOq, is the angle which the object, thus seen, subtends at the center of the eye; and this angle varies as $\frac{pq}{Oq}$*. Now, $QF : FE$ (:: $QE : Eq$) :: $QP : qp$, therefore $qp = \frac{FE \times QP}{QF}$. Also, $QF : FE :: Ef : fq$ (Art. 186); therefore, $fq = \frac{FE \times Ef}{QF}$, and, when Q is between F and E, and O beyond f, $Oq = fq + Of = \frac{FE \times Ef}{QF} + Of = \frac{FE \times Ef + QF \times Of}{QF} = \frac{FE \times Ef + \overline{FE - QE} \times Of}{QF} = \frac{FE \times \overline{Ef + Of} - QE \times Of}{QF} = \frac{FE \times OE - QE \times Of}{QF}$. Consequently, $\frac{pq}{Oq} = \frac{FE \times QP}{FE \times OE - QE \times Of}$; therefore, the visual angle varies as $\frac{FE \times QP}{FE \times OE - QE \times Of}$; or, since FE and QP are invariable, inversely as $FE \times OE - QE \times Of$.

When O is between E and f, Of is negative, and the visual angle varies inversely as $FE \times EO + QE \times Of$.

When the image pq is between O and f, the expression, $FE \times EO - QE \times Of$, becomes negative; for, $Oq = Of - fq = \frac{QE \times Of - FE \times EO}{QF}$; and therefore

* In these, and other calculations of the same kind, the angles contained by the axes of the extreme pencils, at the center of the eye, are supposed to be small; and our conclusions, though not strictly true, are sufficiently accurate for the purposes to which they are applied.

$\frac{pq}{Oq} = \frac{FE \times QP}{QE \times Of - FE \times EO}$; and the visual angle varies inversely as $QE \times Of - FE \times EO$. This shews that the angle pOq lies on the other side of the axis



QO; and the image upon the retina, which was before inverted, will now be erect.

(284.) Cor. 1. When the eye is placed close to the glass, the expression, $FE \times EO + QE \times Of$, becomes $QE \times Ef$; therefore the visual angle varies inversely as QE. In this case, the $\angle pOq =$ the $\angle PEQ$.

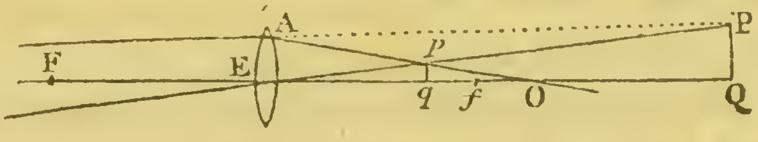
(285.) Cor. 2. When O coincides with f, the expression becomes $FE \times Ef$, which is invariable. That is, when the eye is in the principal focus of the glass, the visual angle is the same, whatever be the distance of the object from the lens.

(286.) Cor. 3. When Q coincides with F, the expression becomes $FE \times EO \pm FE \times Of$, or $FE \times Ef$. That is, the visual angle is the same, whatever be the distance of the eye from the glass; and it is equal to the visual angle when the eye is in the principal focus, and to the angle which the object subtends at the center of the lens.

(287.) Cor. 4. When the eye is farther from the glass than the principal focus f, as QE decreases, $FE \times EO - QE \times Of$ increases; and, therefore, the visual angle decreases, unless the image fall between the eye and the glass; in which case the visual angle varies inversely as $QE \times Of - FE \times EO$; and therefore it increases as QE decreases.

(288.) Cor. 5. When the eye is between the principal focus and the lens, as QE decreases, the expression $FE \times EO + QE \times Of$ decreases; and, therefore, the visual angle increases.

(289.) Cor. 6. When the rays, tending to form the image QP, are intercepted by the glass, and after-

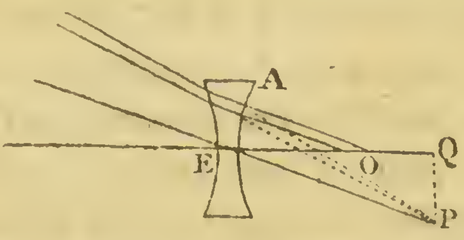


wards received by the eye, QE becomes negative; and the visual angle, when EO is greater than Ef, varies inversely as $FE \times EO + QE \times Of$. When EO is less than Ef, the visual angle varies inversely as $FE \times EO - QE \times Of$, or $QE \times Of - FE \times EO$, according as EO is greater, or less than Eq.

Prop. LXII.

(290.) *The rays which, after reflection or refraction, tend to form an image, may be refracted to the eye by a concave lens, in such a manner as to form a distinct image upon the retina.*

Let the rays which tend to form the image PQ, be received upon the concave lens AE*, whose focal



length is EQ. Then, since P is the principal focus

* Of this description are the double concave, the plano concave, and the concavo convex lenses.

of the lens, the rays which converge to *P* will, after refraction, be parallel to each other (Art. 169), and therefore proper for vision to common eyes; that is, a distinct image of *P* will be formed upon the retina. In the same manner it appears, that a distinct image of every other point in *PQ* will be formed upon the retina; and thus a complete, and distinct image of the whole object will be formed there.

If the eye require diverging rays, the glass must be moved farther from the image *PQ*. For, then the rays in each pencil converge to a point farther from the glass than the principal focus; and therefore, after refraction they will diverge (Art. 225), and be proper for vision in this case.

If the eye require converging rays, the lens must be moved the contrary way.

Prop. LXIII.

(291.) *When the image* QP *is in the principal focus of the concave lens, the picture upon the retina is erect with respect to* QP.

Let *E* be the center of the lens *; *O* the place of the eye; *P* the lowest point in the image; join *PE*; and draw *AO* parallel to *EP*. Then, those rays of the pencil belonging to *P*, which are received by the eye, enter the pupil in the direction *AO*, and proceed to the lower part of the retina; and those which belong to *Q*, enter the pupil in the direction *EO*, and proceed to the upper part; consequently, the picture upon the retina is erect with respect to *QP*.

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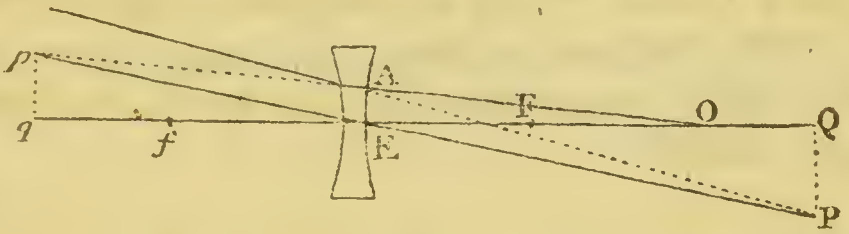
* See the last Figure.

(292.) COR. In the same manner it appears, that when QP is *nearly* in the principal focus, and the eye not very distant from the glass, the image upon the retina is erect.

PROP. LXIV.

(293.) *To find how the visual angle varies, when rays, which tend to form an image in the axis of a concave lens, are refracted to an eye situate in that axis.*

Let rays which tend to form the image QP be intercepted by the concave lens AE, whose center is E, and axis qFQ; and after refraction let them be received



by an eye placed at O. Take qp the image of QP, and join pO. Then, the visual angle pOq varies as $\frac{qp}{Oq}$. Now, $QF : FE$ (:: $QE : Eq$) :: $QP : qp$; therefore, $qp = \frac{QP \times FE}{QF}$. Also, $QF : FE :: Ef : fq$; whence, $fq = \frac{FE \times Ef}{QF}$; and, when EQ is greater than EF, $Oq = fq + Of$ (Art. 188) $= \frac{FE \times Ef + QF \times Of}{QF} =$

$$\frac{FE \times Ef + \overline{QE - FE} \times Of}{QF} = \frac{FE \times \overline{Ef - Of} + QE \times Of}{QF}$$

$= \frac{QE \times Of - FE \times EO}{QF}$; therefore, $\frac{qp}{Oq} =$

$\dfrac{QP \times FE}{QE \times Of - FE \times EO}$; and since QP and FE are given, the visual angle varies inversely as $QE \times Of - FE \times EO$.

(294.) Cor. 1. When the eye is placed close to the glass, EO vanishes; therefore the visual angle varies inversely as $QE \times Of$; that is, inversely as QE. In this case, the angle qOp becomes equal to QEP.

(295.) Cor. 2. When Q coincides with F, the expression, $QE \times Of - FE \times EO$, becomes $FE \times \overline{Of - EO} = FE \times Ef$, which is invariable; in this case, therefore, the visual angle is the same, whatever be the distance of the eye from the glass.

(296.) Cor. 3. When the expression, $QE \times Of - FE \times EO$, becomes negative, the image upon the retina, which was before erect, will be inverted with respect to QP.

On the Astronomical Telescope.

(297.) The astronomical telescope consists of two convex lenses, whose axes are in the same right line, and whose distance is equal to the sum of their focal lengths.

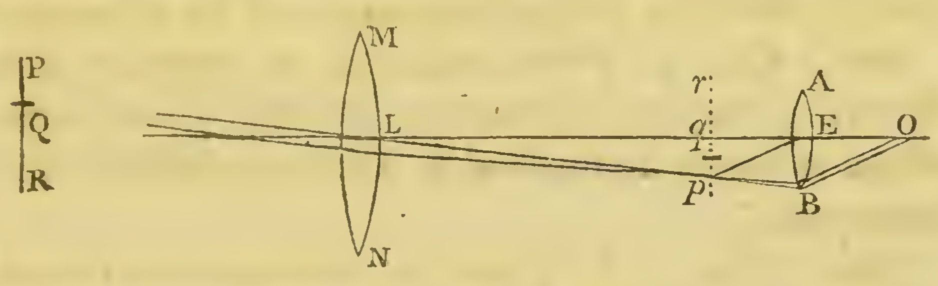
(298.) The common axis of the two lenses is called the *axis* of the telescope.

(299.) The lens which is turned towards the object to be viewed, and which has the greater focal length, is called the *object glass*. The lens to which the eye is applied, is called the *eye glass*.

Prop. LXV.

(300.) *A distant object may be seen distinctly through the astronomical telescope; and the angle which it subtends at the center of the eye, when thus seen, is to the angle which it subtends at the naked eye, as the focal length of the object glass, to the focal length of the eye glass.*

Let L and E be the centers of the two glasses; QP an object, towards which the axis of the telescope is directed; and so distant, that the rays which flow from any one point in it, and fall upon the object glass L,



may be considered as parallel. Then qp, an inverted image of QP, will be formed in the principal focus of the glass L, and contained between the lines QLq, and PLp (Art. 214); and, because LE is equal to the sum of the focal lengths of the two glasses, pq is in the principal focus of the glass AB; consequently, pq may be seen distinctly through this glass, if the eye of the observer be able to collect parallel rays upon the retina (Art. 280). Produce PLp till it meets the eye glass in B; join pE; and draw BO parallel to pE. Then, the rays which flow from P in the object, or p it's image, enter the eye, placed at O, in the direction BO. Also, the rays which flow from Q, enter the eye in the direction EO; thus, the angle which QP subtends at the center of the eye, when viewed

through the telescope, is the angle BOE, which is equal to pEq. The angle which QP subtends, when viewed with the naked eye from L, is PLQ, which is equal to pLq. And, since the $\angle pEq$: the $\angle pLq$ (when these angles are small)* :: $Lq : Eq$, the angle which the object subtends at the center of the eye, when seen through the telescope : the angle which it subtends at the center of the naked eye :: $Lq : Eq$.

(301.) Cor. 1. If the angle which the object subtends at the center of the naked eye be given, the angle which it subtends at the center of the eye, when seen through the telescope, varies as $\frac{Lq}{Eq}$. This quantity is usually called the *magnifying power* of the telescope.

(302.) Cor. 2. If the telescope be inverted, the object may be seen distinctly; and the visual angle will be as much diminished as it was magnified in the former case.

(303.) Cor. 3. To adapt the telescope to a nearer object, the eye glass must be moved farther from the object glass.

For, if QP be brought nearer to the glass L, the image qp will be formed at a greater distance from it (Art. 189); and therefore, in order that qp may still be in the principal focus of the glass E, this glass must be moved farther from L.

(304.) Cor. 4. When QP is brought nearer to L, the magnifying power is increased. For, Lq is increased, and Eq remains the same; therefore $\frac{Lq}{Eq}$ is increased.

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* Proportions of this kind are to be considered only as approximations, which become more accurate as the angles decrease.

(305.) Cor. 5. To adjust the telescope to the eye of a short sighted person, the eye glass must be moved nearer to the object glass. If the eye require converging rays, the eye glass must be moved the contrary way (Art. 280).

(306.) Cor. 6. If we suppose the eye to be placed between the glass AB and it's principal focus, the visual angle is increased by adjusting the telescope to the eye of a short sighted person (Art. 288). If the eye be farther from the glass than it's principal focus, that angle is diminished (Art. 287). The contrary effects are produced when the telescope is adjusted to the eye of a long sighted person.

Prop. LXVI.

(307.) *Objects, seen through the astronomical telescope, appear inverted.*

The image of pqr is inverted upon the retina (Art. 281); and pqr is inverted with respect to PQR (Art. 221); therefore the image upon the retina is erect with respect to PQR; and consequently the object appears inverted (Art. 249).

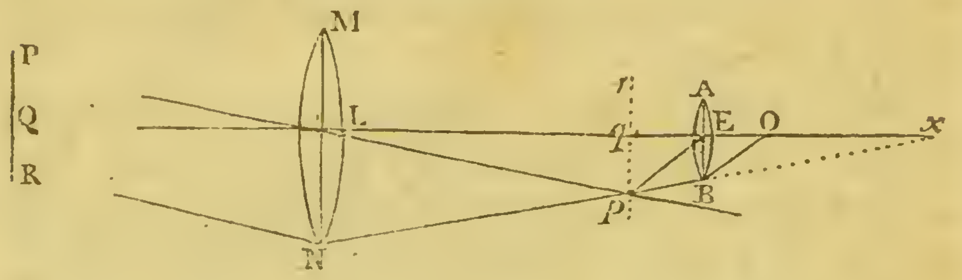
(308.) Cor. An object moving across the field of the telescope from the right to the left, appears to move from the left to the right.

Prop. LXVII.

(309.) *In the astronomical telescope, the field of view is the greatest, when the eye is placed between the eye glass and it's principal focus.*

The field of view is the greatest, when the eye is so placed as to receive the extreme rays, refracted from

the object glass to the eye glass. Let MN be the diameter of the object glass; AB, the diameter of the

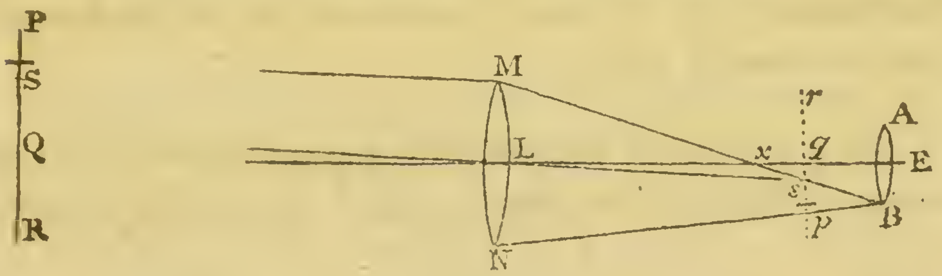


eye glass; join N, B, the corresponding extremities of the glasses, and let NB cut the image pqr in p; join pL, and suppose pL, EL to be produced till they meet the object QP, in P and Q; join also, pE, NP; and draw BO parallel to pE. Then, if the eye be placed at O, it will receive the ray NBO. And this is the ray which comes from a point in the object, at the greatest visible distance from the axis of the telescope; for, the rays which flow from any point in the object, above P, will, after the first refraction, converge to a point below p, and all of them fall below the glass AB; therefore P is the extreme point visible in the object, and QP is half the linear magnitude of the visible area. Now, when the diameter of the object glass is greater than that of the eye glass, which is the case in the astronomical telescope, the rays NB, LE converge to some point x beyond the eye glass; and therefore, the point O to which they converge after the second refraction, and where the eye must be placed to receive them, is between the glass E and it's principal focus (See Arts. 186. 188.)

(310.) Cor. 1. If NB and LE be parallel, O is the principal focus of the glass E; and if they diverge, O lies beyond the principal focus.

(311.) Cor. 2. It appears from the preceding figure, that NBO is the only ray which comes to the eye from P; for, any other ray of the pencil, as PLp, after refraction at the object glass, crosses NB in p, and falls below the eye glass. Hence it follows, that the extremity of the visible area is very faint. If a point be taken in the object, nearer to the center of the field of view, more of the rays which flow from it will be refracted to the eye; and thus, the brightness will continually increase till all the rays in each pencil, incident upon MN, are received by the glass AB; when this takes place, the brightness will become uniform; because the same number of rays, or very nearly so, is received by the glass MN from every point in the object *.

(312.) Cor. 3. To determine the bright part of the field of view, join MB, cutting pqr in s, and the axis LE in x; join also sL, and produce it till it meets the



object in S. Then, if xE be greater than qE, all the rays which flow from S, and fall upon MN, will be refracted to the eye glass; for, SM is refracted in the direction MsB; and any other ray of the pencil, as SL, crosses MB at s, and falls somewhere between A

* Here it is supposed that the pupil is properly placed, and so large as to receive all the rays refracted by the lens AB.

and B. In the same manner, the rays which flow from any point between S and Q, and fall upon MN, will be refracted to the eye glass AB.

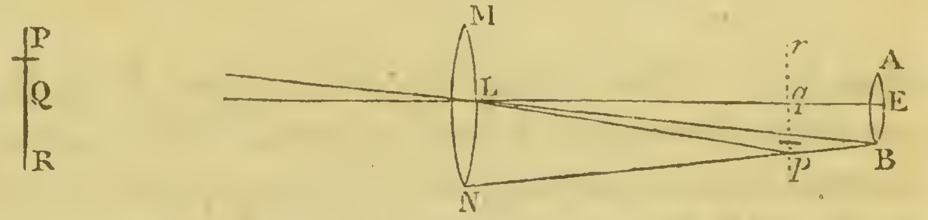
(313.) Cor. 4. If q and x coincide, that is, if the linear apertures, MN, AB, of the glasses, be proportional to their focal lengths, the brightness of the field increases to the center.

(314.) Cor. 5. If q fall between L and x, the whole of the rays belonging to any one pencil incident upon MN, will not be received by the eye glass. In this case, a less aperture of the object glass would produce the same brightness, with less aberration*.

Prop. LXVIII.

(315.) *The linear magnitude of the greatest visible area is measured by the angle which the diameter of the eye glass subtends at the center of the object glass, increased by the difference between the angles which the diameter of the object glass subtends at the image, and at the eye glass.*

Let pqr, as in the preceding propositions, be the image formed by the object glass. Join NB, cutting



pqr in p; join also, LB, Lp; and suppose pL, EL to be produced till they meet the object in P and Q. Then, QP, which is half the linear magnitude of the

* See Sect. viii.

greatest visible area (Art. 309), is measured by the angle PLQ, or it's equal pLq; that is, by $BLE + BLp$; or, $BLE + LpN - LBN$; therefore, $2\,QP$ is measured by $2\,BLE + 2\,LpN - 2\,LBN$.

(316.) Cor. In the same manner it may be shewn, that the linear magnitude of the brightest part of the visible area, is measured by the angle which the diameter of the eye glass subtends at the center of the object glass, diminished by the difference between the angles which the diameter of the object glass subtends at the image, and at the eye glass.

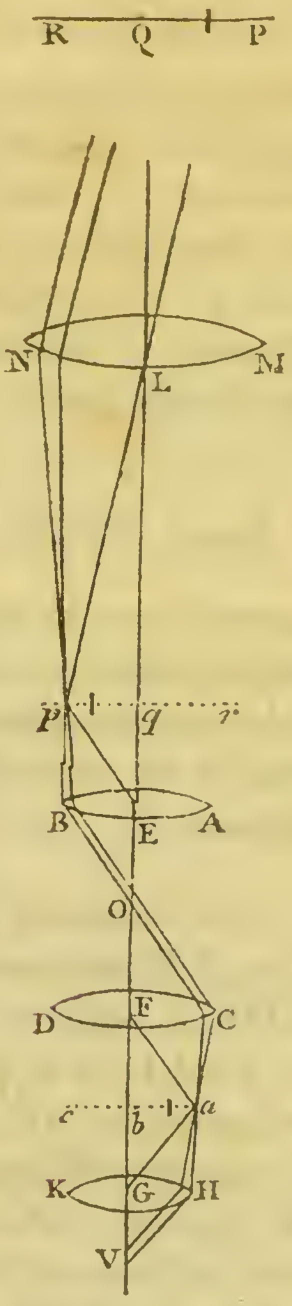
Prop. LXIX.

(317.) *If two convex lenses be added to the former, and placed in a similar manner, a distant object may be seen distinctly through the telescope, and the visual angle will be altered in the ratio of the focal lengths of the additional glasses.*

Let CD, HK, be the additional glasses, whose axes coincide with the line LE produced, and the distance of whose centers, FG, is equal to the sum of their focal lengths; also, let CD be so placed as to receive the extreme rays refracted by AB.

Then, since the rays in each pencil, after refraction at the glass E, are parallel (Art. 300), they will be collected in the principal focus of the glass CD; that is, in the principal focus of the glass HK; and an image, abc, will be formed there, which may be seen distinctly through the lens HK (Art. 280).

Again, let *BC* be the extreme pencil of rays refracted by *AB*; draw *Fa* parallel to *BC*, and let it meet *abc*



in *a*; join *aG*, *Ca*; and produce *Ca* till it meets the lens *HK* in *H*; draw *HV* parallel to *aG*; and at *V* let the eye be placed. Then, *NBCHV* being the course of the pencil of rays which flows from *P*; and *LEFGV* the course of the pencil which flows from *Q*,

the angle which QP subtends at the center of the eye, when seen through the four glasses : the angle which it subtends there, when seen through the first two :: the $\angle HVG$: the $\angle EOB$ :: the $\angle aGb$: the $\angle aFb$ :: $Fb : Gb$.

(318.) Cor. 1. Since the angle which QP subtends at the center of the eye, when it is seen through the two first glasses : the angle which it subtends at the center of the naked eye :: $Lq : Eq$ (Art. 300), by compounding this proportion with the last, we have, the angle which the object subtends at the center of the eye, when viewed through the four glasses : the angle which it subtends at the naked eye :: $Lq \times Fb$: $Eq \times Gb$.

(319.) Cor. 2. If Gb and Fb be equal, the visual angle is not altered.

(320.) Cor. 3. The object, when viewed through the glasses thus combined, appears erect. For, the axes of the several pencils of rays which flow from the image pqr, cross each other at E; therefore, the image abc is inverted with respect to pqr; or, erect with respect to the object PQR; and consequently, the image upon the retina is inverted (Art. 281); that is, the object appears erect*.

(321.) Cor. 4. If the apertures of the additional glasses be properly adjusted, the field of view will not be altered. For, the extreme rays, refracted by AB, may be received by CD, and refracted to HK.

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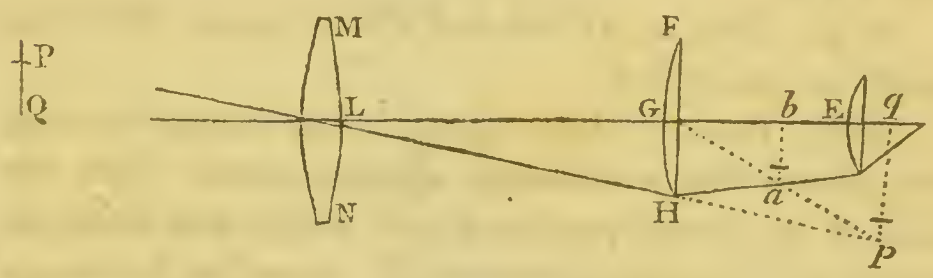
* This is one advantage gained by the additional glasses. The inconvenience with which they are attended is, that they render objects more faint, by increasing the quantity of the refracting medium through which the rays must pass, before they arrive at the eye (See Note, p. 137.)

(322.) Cor. 5. This telescope may be adjusted to the eye of a short sighted person, by moving the glass *HK* nearer to *CD* (Art. 280); or, if *E, F, G,* be connected, by moving these three glasses nearer to *L*. If the eye of the observer require converging rays, the glasses must be moved the contrary way.

(323.) Sometimes a convex lens is interposed between the object glass and it's principal focus, in the astronomical telescope, for the purpose of increasing the field of view, and diminishing the aberration of the lateral rays (Sect. viii).

Let *FGH* be such a lens, whose axis is coincident with the axis of the telescope. Then, the rays which



tend to form the image *qp*, after refraction at the glass *FH*, form the image *ab*, between *G* and *q* (Art. 223); and this image is viewed through the eye glass *E*, whose focal length is *bE**.

On Galileo's Telescope.

(324.) Galileo's telescope consists of a convex and a concave lens, whose axes are in the same line, and whose distance is equal to the difference of their focal lengths.

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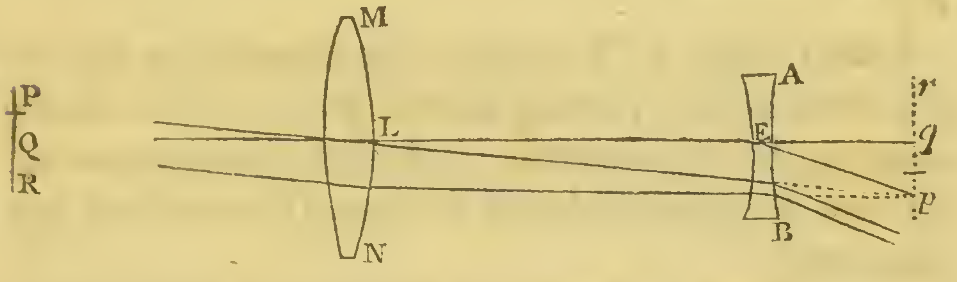
* Other constructions, with the explanation of their advantages, may be seen in the *Encyclopædia Britannica*, Article *Telescope*.

Prop. LXX.

(325.) *A distant object may be seen distinctly through Galileo's telescope; and the angle which it subtends at the center of the eye when thus seen, is to the angle which it subtends at the center of the naked eye, as the focal length of the object glass, to the focal length of the eye glass.*

Let L and E be the centers of the glasses; PQR a distant object, towards which the axis of the telescope



is directed; pqr it's image in the principal focus of the glass L, and therefore in the principal focus of the glass E; then, since the rays tend to form an image in the principal focus of the concave lens E, after refraction at that lens, they will be proper for vision (Art. 290); or, a distinct image of the object PQR, will be formed upon the retina of a common eye.

Also, the angle under which the object QP is seen through the telescope, is equal to the angle qEp (Art. 294); and the angle under which it is seen with the naked eye from L, is QLP, which is equal to qLp; therefore, the visual angle in the former case : the visual angle in the latter :: Lq : Eq.

(326.) Cor. 1. The magnifying power of the telescope is measured by $\frac{Lq}{Eq}$ (See Art. 301).

(327.) Cor. 2. To adapt this telescope to a nearer object, the eye glass must be moved to a greater distance from the object glass.

For, as QL decreases, Lq increases (Art. 189); therefore, in order that the principal focus of the glass E may coincide with q, LE must be increased.

This is the construction of a common opera glass.

(328.) Cor. 3. When the telescope is adjusted to a nearer object, the magnifying power is increased; for Lq is increased, and Eq remains the same; therefore $\frac{Lq'}{Eq}$ is increased.

(329.) Cor. 4. To adjust this telescope to the eye of a short sighted person, the eye glass must be moved nearer to the object glass; if the eye require converging rays, the eye glass must be moved the contrary way (Art. 290).

(330.) Cor. 5. To take in the greatest field of view, the eye must be placed close to the glass AB, as will be shewn in a subsequent article; and therefore, by adjusting the telescope to the eye of a short sighted person, the visual angle of a given object is diminished (Art. 294); and on the contrary, by adjusting it to the eye of a long sighted person, this angle is increased.

Prop. LXXI.

(331.) *Objects, seen through Galileo's telescope, appear erect.*

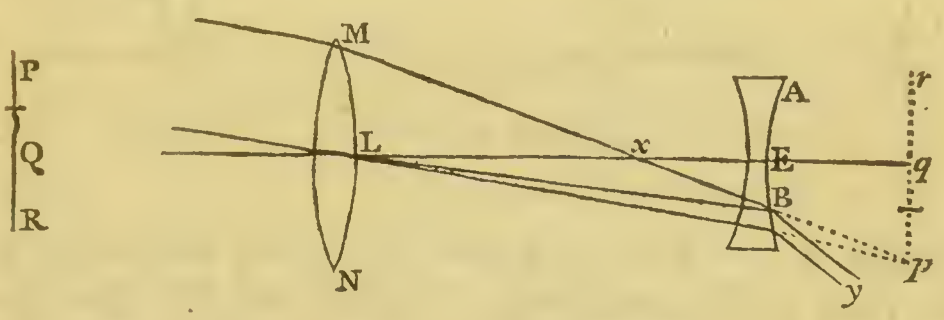
For, the image upon the retina is erect, with respect to pqr (Art. 291); therefore it is inverted

with respect to the object; or the object appears erect (Art. 249).

Prop. LXXII.

(332.) *The linear magnitude of the field of view, when the eye is placed close to the concave lens in Galileo's telescope, is measured by the angle which the diameter of the pupil subtends at the center of the object glass, increased by the difference between the angles which the diameter of the object glass subtends at the pupil, and at the image.*

Let MN be the diameter of the object glass; AB, the diameter of the pupil, whose center is in the axis



of the telescope; join M, B, the opposite extremities of these diameters; and let MB meet the axis in x, and the image in p; join LB, Lp; and suppose pL, qL to be producėd till they meet the object in P and Q; also, suppose MP to be joined. Then will PM be refracted to the pupil in the direction MB; but any other ray in the pencil, as PL, and every ray which flows from a point more distant from the axis of the telescope, and is refracted by MN, will fall below the pupil; therefore QP is half the linear magnitude of the greatest visible area. Also, QP is measured by the angle QLP, or it's equal $pLq = ELB +$

$BLp = ELB + LBM - LpM$; and by doubling these quantities, $2QP$ is measured by $2ELB + 2LBM - 2LpM$.

(333.) Cor. 1. The rays MxB, LxE, which are incident upon the glass E, diverging from x, after refraction will diverge more (Art. 190); therefore, if the eye be moved to any other point in the axis of the telescope, the ray xBy will not enter the pupil; and consequently the visible area will be diminished.

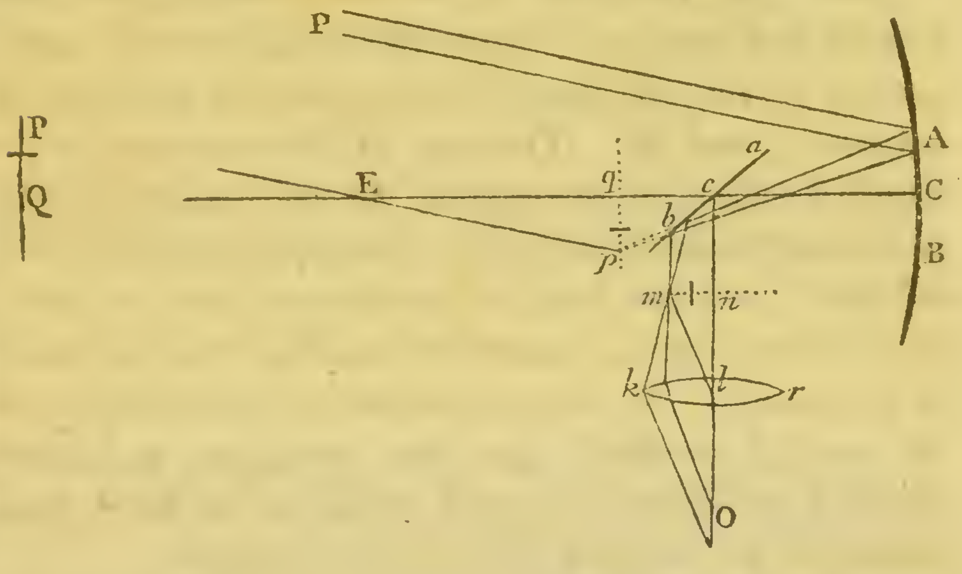
(334.) Cor. 2. To determine the brightest part of the visible area, join NB, and let it meet the image in s; join BL, sL, and produce sL till it meets the

object in S. Then, all the rays which flow from S, or from any point between S and Q, and fall upon the object glass, will be refracted to the eye*; and QS will be the linear magnitude of half the bright part of the field of view. Also, QS is measured by the angle SLQ, which is equal to $sLq = BLE - BLs = BLE - LBN + LsN$; and $2QS$ is measured by $2BLE - 2LBN + 2LsN$.

* See Art. 311.

On Sir Isaac Newton's *Telescope.*

(335.) Let ACB be a concave spherical reflector, whose middle point is C, and center E. Join CE,



this is called the *axis of the reflector*, or of the telescope. Let CE be directed to the point Q, in the distant object QP; then, qp, an inverted image of QP, would be formed in the principal focus of the reflector, at right angles to CE (Art. 47), and terminated by the lines PEp, QEq, were the reflected rays suffered to proceed thither; but, before they arrive at the focus, they are received upon a plane reflector acb, inclined at an angle of 45° to the axis CE; and thus an image, mn, is formed, similar and equal to pq, and equally inclined to the plane reflector (Art. 72); and consequently, mn is parallel to EC. This image is viewed through a convex eye glass klr, whose axis is perpendicular to EC, and whose distance from the image mn, is equal to it's focal length*.

* See Art. 280.

If the reflection be made at c, by a small right angled prism, one of whose sides is perpendicular, and the other parallel to the axis, much less light will be lost, than if the reflection be made by a plane speculum (See Art. 101).

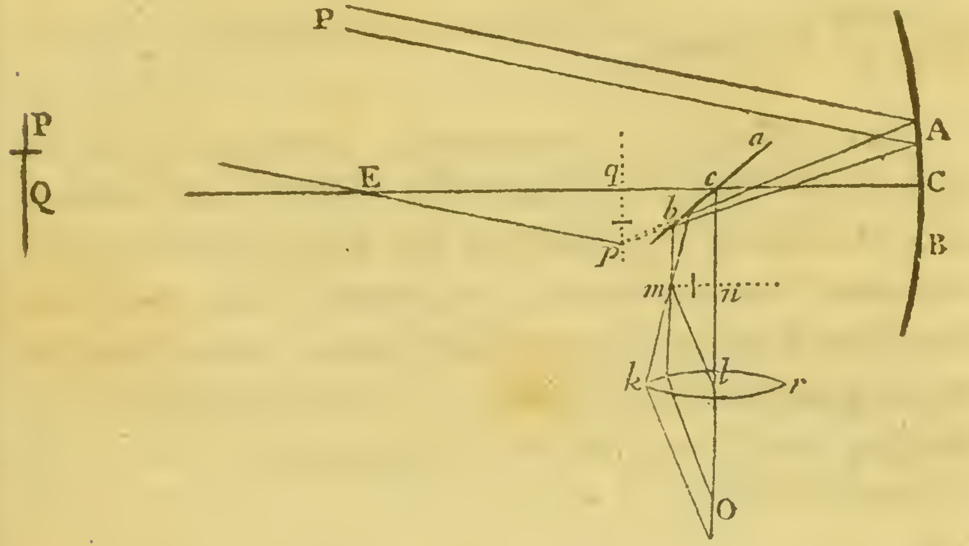
(336.) Dr. Herschel has so far increased the focal lengths and apertures of his reflectors, that the image qp can be viewed directly, through an eye glass placed between q and E. The axis of the telescope is inclined a little from the object, that the image qp may be formed near the side of the tube which contains the reflector; and the head of the observer does not intercept so many rays as materially to affect the brightness of the image. By this construction, one reflection of the rays is avoided; and the strongest, and most effective pencils are preserved, which, in the Newtonian telescope, are stopped by the plane reflector.

Prop. LXXIII.

(337.) *When a distant object is viewed with Sir Isaac Newton's reflecting telescope, the angle which it subtends at the center of the eye, is to the angle which it subtends at the naked eye, as the focal length of the reflector, to the focal length of the eye glass.*

Let QP be the object; qp it's image, in the principal focus of the reflector; QEC the axis of the telescope, cutting acb in c. Draw $cnlO$ perpendicular to CE; make cn equal to cq, draw nm at right angles to cn, and make nm equal to qp; then nm is the image of qp, or QP. At l let a convex eye glass klr be placed, whose focal length is ln, and whose

axis coincides with that line; join ml. Then the image nm, which corresponds to QP in the object, is



seen through the glass klr, under an angle which is equal to mln; and QP is seen, with the naked eye placed at E, under an angle which is equal to qEp; and since these angles have equal subtenses nm, and qp, they are to each other inversely as the radii ln, qE; therefore the angle which the object subtends at the center of the eye, when viewed with the telescope : the angle which it subtends at the naked eye :: $Eq : ln$.

(338.) Cor. 1. The magnifying power of this telescope is measured by $\frac{Eq}{ln}$.

(339.) Cor. 2. To adapt the telescope to nearer objects, the reflector acb, to which the eye glass is attached, must be moved towards E. For, as QE decreases, qE decreases (Art. 59); and therefore, that qc, or cn may remain of the same magnitude, acb must be moved nearer to E.

(340.) Cor. 3. As the object is brought nearer to E, the magnifying power of the telescope is diminished. For, Eq decreases, and ln is invariable; therefore $\frac{Eq}{ln}$ decreases.

(341.) Cor. 4. To adjust the telescope to the eye of a short sighted person, the reflector acb must be moved towards C. For then the image mn will be at a greater distance from c, or nearer to the eye glass; and therefore, the rays in each pencil, after refraction at the glass klr, will diverge. If the eye require converging rays, the reflector acb must be moved the contrary way.

Prop. LXXIV.

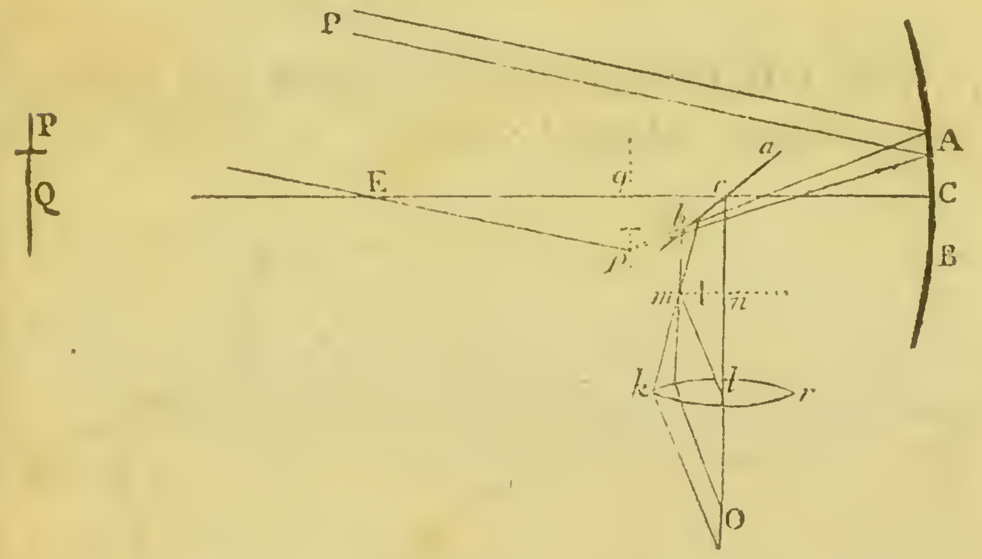
(342.) *Objects, viewed with Sir Isaac Newton's telescope, appear inverted.*

Let QP lie in the plane of the paper; and, when the eye of the observer is applied to the glass kr, let the plane which passes through both his eyes, also coincide with the plane of the paper. Then, the rays which flow from P, the right side of the object, converge to m, the left side of the image. Also, the rays which flow from a point in the object, above the plane of the paper, converge to a point in the image, below that plane; thus, the image mn is inverted; and since it is in, or near to, the principal focus of the convex lens through which it is viewed, it appears inverted (Art. 281).

Prop. LXXV.

(343.) *To determine the field of view in Sir Isaac Newton's telescope.*

Join b, k, the corresponding extremities of the plane reflector and the eye glass; and let bk cut the image



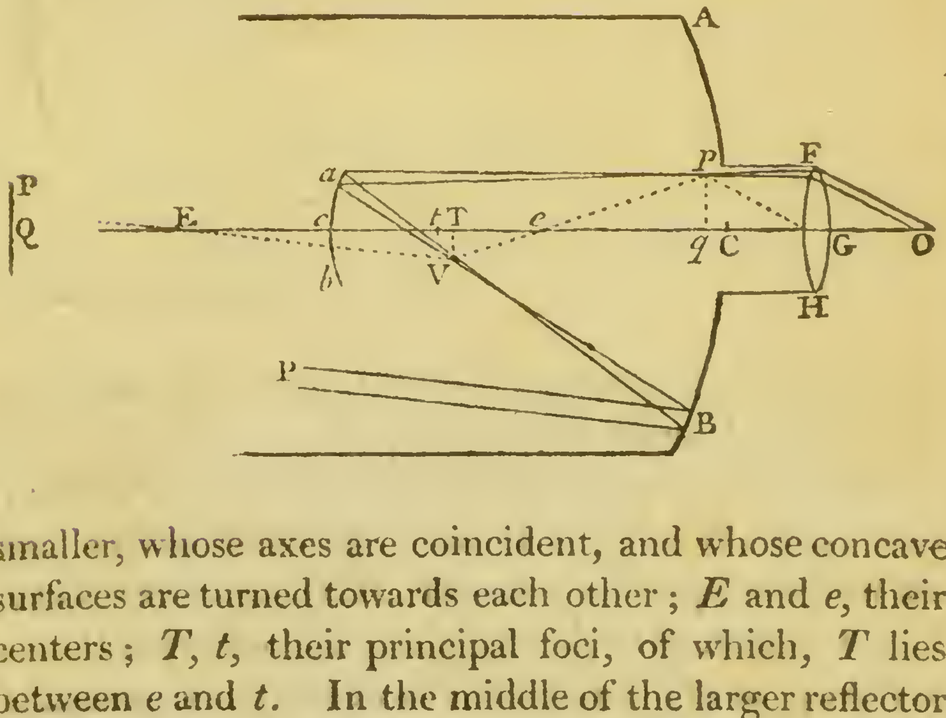
mn in m: take qp equal to nm; draw pE, and produce it till it meets the object in P.

Then, a small pencil of rays flowing from P, will, if acb be not so large as to intercept them before they are incident upon the reflector ACB, be reflected at b, to the extremity of the eye glass, and refracted thence to the eye at O; the point P will therefore be visible. Also, this point is the extremity of the field of view. For, if a point be taken in nm, farther from the axis of the lens than m, any straight line drawn through it, will either fall without ab, or without kr; that is, no ray, belonging to a point in the object QP, beyond P, can be reflected from acb to the eye glass.

(344.) Cor. In order that the field of view may be circular, *acb* must be the transverse section of a cone, or cylinder, generated by the revolution of *b k*, about the axis *c l*; and therefore it must be an ellipse, whose major and minor axes depend upon the nature of this solid.

On the Gregorian *Telescope.*

(345.) In the annexed figure, *ACB*, *acb* represent two concave spherical reflectors, a greater and a



smaller, whose axes are coincident, and whose concave surfaces are turned towards each other; *E* and *e*, their centers; *T*, *t*, their principal foci, of which, *T* lies between *e* and *t*. In the middle of the larger reflector is an opening, nearly of the same dimensions with the aperture of the smaller, to admit a moveable tube containing a convex eye glass *FGH*.

When the axis of the telescope is directed to the point *Q* in a distant object *QP*, an inverted image *TV*, of this object, terminated by the lines *QET*, *PEV*, is formed in the principal focus of the reflector

AB. The rays which diverge thence, and fall upon the concave reflector *acb*, after reflection, form an image *qp*, terminated by the lines *Teq*, *Vep*; which, because *T* is between *e* and *t*, is inverted with respect to *TV* (Art. 88); or erect, with respect to *QP*.

The rays are then received by the eye glass *FGH*, whose axis coincides with the axis of the telescope, and whose focal length is *Gq*; and therefore the image *qp* may be seen distinctly.

(346.) If the distance *Cc* be diminished, the distance *tq* will be increased.

For, as *Cc* decreases, *Tt* decreases, and $Tt : te :: te : tq$, in which proportion *te* is invariable; therefore *tq* increases*. By a proper adjustment then, of the reflectors and eye glass, the image *qp* may be formed in the principal focus of the lens *FGH*; or in such a situation as may be necessary for distinct vision (See Art. 280).

(347.) This telescope may be adapted to nearer objects, by increasing the distance *Cc*. For, as *QP* approaches towards *E*; *TV* also approaches towards *E*, or towards *t*; therefore the distance *tq* increases; or *q* is nearer to the eye glass than before; which inconvenience may be remedied by increasing the distance *Cc* (Art. 346).

(348.) Objects seen through this telescope, appear erect. For, *qp* is an erect image of *QP*, and in, or

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* Having given the focal lengths of the reflectors, and the distance of *q* from *T*, the distance of the reflectors may be found.

Let $Tt=x$; $te=b$; $TC=a$; $Tq=c$. Then, $x : b :: b : x+c$; therefore, $x^2+cx=b^2$; from which equation we obtain $x=\frac{\sqrt{4b^2+c^2}-c}{2}$; and $Cc=a+b+x$.

near to the principal focus of the convex lens through which it is viewed; therefore it appears erect (Art. 281).

(349.) As so much has been said upon the field of view in other telescopes, the reader will find no difficulty in determining it in this case. Join F, a, corresponding extremities of the small reflector, and the eye glass; let Fa cut qp, in p. Draw pe, and produce it till it meets TV in V; join VE, and produce this line till it meets the object QP in P; then is P the extremity of the field of view. For, if aVB be drawn, the rays which flow from P, and fall upon the large mirror at B, after reflection converge to V, and fall upon the reflector acb at a; thence they proceed in the direction apF, and are refracted to the eye in the direction FO, which is parallel to pG*. But no ray, which belongs to a point in the object above P, can be reflected from acb to the eye glass FH.

Since qp is much nearer to the eye glass than to the reflector acb, the field of view will depend more upon the aperture of the former, than of the latter.

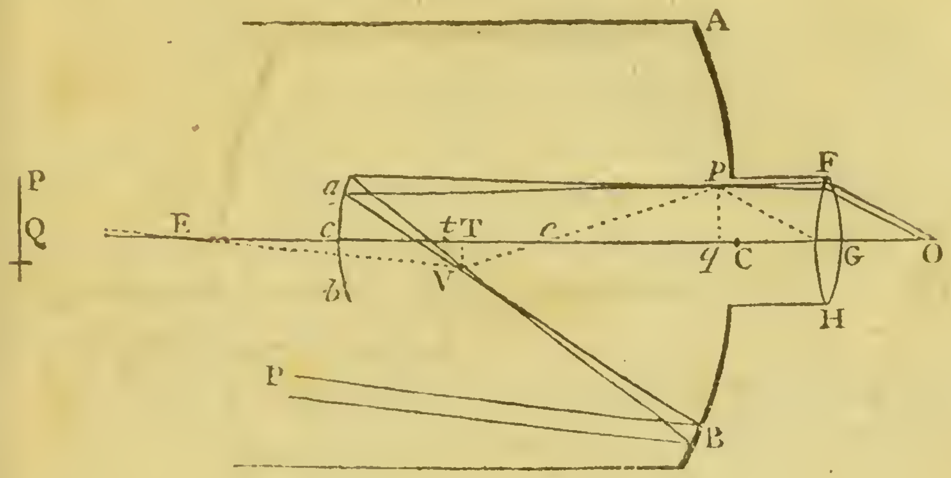
Prop. LXXVI.

(350.) *To determine the angle which an object subtends at the center of the eye, when seen through Gregorie's telescope.*

The construction being made as before, the angle under which QP is seen through the telescope, is

* This is on supposition that aVB meets the reflector AB; and that the ray PB, which is parallel to EV, is not intercepted by the reflector ab.

equal to pGq; and the angle under which it is seen with the naked eye, is equal to TEV. Now, when



the angles are small, the $\angle pGq$: the $\angle peq$ (TeV) :: $eq : qG$; and also, the $\angle TeV$: the $\angle TEV$ :: $ET : eT$; by comp. the $\angle pGq$: the $\angle TEV$ :: $eq \times ET : qG \times eT$; that is, the visual angle, when the object is seen through the telescope : the visual angle, when it is seen with the naked eye :: $eq \times ET : qG \times eT$.

(351.) Cor. Since $Tt : te :: eT : eq$ (Art. 54); inversely, $te : Tt :: eq : eT$; therefore $te \times ET : Tt \times qG :: eq \times ET : qG \times eT$ (Alg. Art. 185); and the visual angle, when the object is seen through the telescope : the visual angle, when it is seen with the naked eye :: $te \times ET : Tt \times qG :: \frac{te \times ET}{qG} : Tt$.

On Cassegrain's *Telescope*.

(352.) In this telescope, the smaller reflector, acb, is convex, and so placed, that T falls between t and c. In other respects, it is similar to the Gregorian telescope (See Art. 345).

Let GCE, the axis of this telescope, be directed to the point Q in a distant object QP; then, an inverted

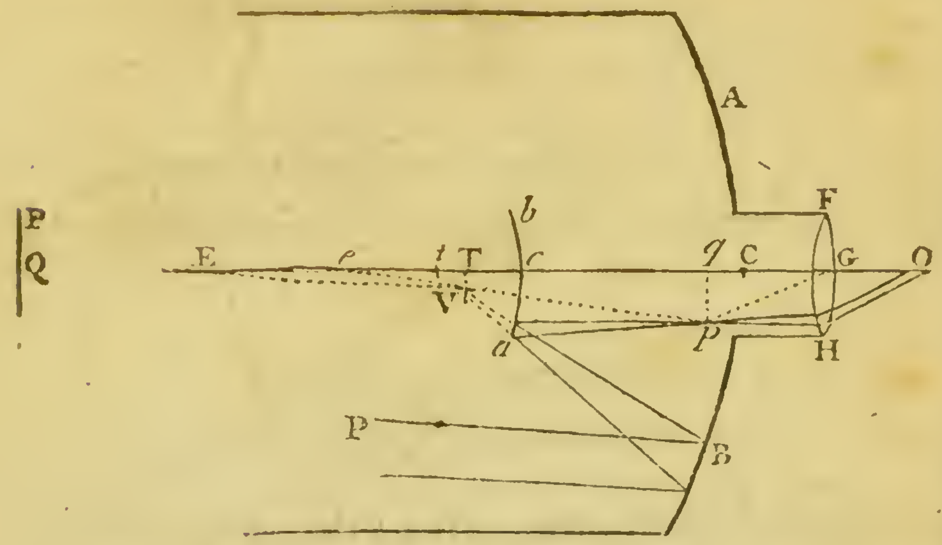


image TV, of this object, would be formed in the principal focus of the large reflector, and terminated by the lines QET, PEV, if the rays were suffered to proceed thither. But, before they reach the focus, they are received upon the convex reflector bca; and since, by the construction, TV falls between the surface c, and principal focus t, of this reflector, an erect image qp, of TV, is formed, and terminated by the lines eTq, eVp. This image is viewed through the lens FGH, whose axis coincides with the axis of the telescope, and whose focal length is Gq.

(353.) The distance of the image qp, from t, may be determined by the proportion $Tt : et :: et : tq$. Also, by diminishing the distance Cc, the distance Tt is diminished, and since te is given, tq is increased.

Hence it is manifest, that by a proper adjustment of the reflectors and the eye glass, the image qp may be formed in the principal focus of the lens FGH;

or in such a situation, that the emergent rays may have a proper degree of divergency, or convergency, for the eye of the observer.

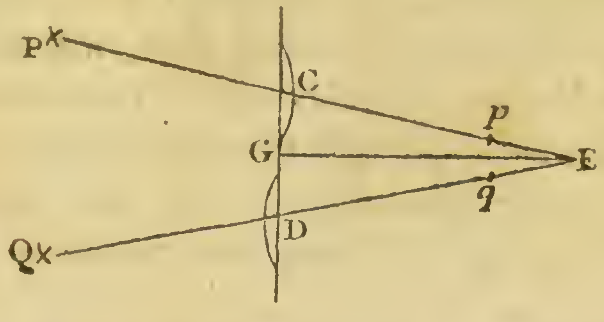
(354.) Objects viewed through this telescope appear inverted. For, the image *qp* is erect with respect to *TV*, or inverted with respect to the object; and it is in, or near to the principal focus of the convex glass through which it is viewed; therefore the image upon the retina is erect, or the object appears inverted (Art. 281).

(355.) The field of view may be determined exactly in the same manner as in the Gregorian telescope; and the demonstration given in Art. 349, may be applied, in the same words, to the preceding figure.

This telescope may also be adapted, in the same manner, to nearer objects; and the visual angle is expressed in the same terms (See Arts. 347. 350).

On the divided object glass Micrometer.

(356.) This micrometer consists of a convex lens divided into two equal parts, *C*, *D*, by a plane which passes through it's axis; and the segments



are moveable on a graduated line *CD*, perpendicular to that axis. Let *C*, *D*, be the centers of the segments; and *P*, *Q*, two remote objects, images of

which will be formed in the lines PCE, QDE, and also in the principal foci of the segments (Art. 219). Let the glasses be moved till these images coincide*, as at E; then, the angle PEQ, which the objects subtend at E the principal focus of C, or D, is equal to the angle which CD, the distance of the centers of the two segments, subtends at the same point; and therefore, by calculating this angle, we may determine the angular distance of the bodies P and Q, as seen from E. Draw EG perpendicular to CD; and, because the triangle CED is isosceles, $CG = GD$, and the $\angle CEG =$ the $\angle GED$; also, GD is the sine of the angle GED, to the radius ED; therefore, knowing ED, and GD, the angle GED may be found by the tables; and consequently $2\,GED$, or CED may be determined.

(357.) The angle CED is in general so small, that it may, without sensible error, be considered as proportional to the subtense CD. And being determined in one case by observation, it may be found in any other, by a single proportion.

(358.) If the objects be at a *given finite* distance, the angle PEQ will still be proportional to CD; for, on this supposition, the distance CE, or DE, of either image from the corresponding glass, will be invariable; therefore, the angle CED will be proportional to CD.

The divided object glass is applied both to reflecting and refracting telescopes; and thus small angular distances in the heavens, are measured with great accuracy.

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* Two other images are formed, an image of P by the segment D, and an image of Q by the segment C; but, as CD increases, these images always *recede* from each other.

On single Microscopes.

(359.) If the angle which an object subtends at the center of the eye, when at a proper distance for distinct vision, be diminished beyond a certain limit, the image upon the retina is so small as to convey to the mind only the idea of a single physical point, not distinguishable into parts; respecting which, therefore, no judgment can be formed by the sight, except what relates to it's colour*. If we endeavour to increase the image upon the retina by bringing the object nearer to the eye, the extreme rays which enter the pupil will diverge too much, and the image become confused. If the extreme rays be stopped, to lessen the indistinctness produced by the lateral rays, the image will be indistinct for want of light†. But if the object be placed in the principal focus of a glass spherule, or lens whose focal length is short, it may be seen distinctly; the visual angle, as well as the quantity of light admitted into the eye, will be increased; and thus, the several parts, of what before appeared only as a single point, will be subjected to examination.

These glasses are called *single Microscopes.*

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* The least visible part of an object does not subtend at the center of the eye, an angle much less than 4′. A single, detached object is perceivable under an angle of about 2′. Harris's *Optics*, p. 121. Jurin's *Essay*, in Smith's *Optics*, Art. 164.

† See Note, p. 130.

Prop. LXXVII.

(360.) *The visual angle of an object when seen through a single microscope, is to it's visual angle when viewed with the naked eye at the least distance of distinct vision, as that least distance, to the focal length of the glass.*

Let QP be an object placed in the principal focus of the lens, or spherule AE, whose center is E; LQ

the least distance at which it can be seen distinctly with the naked eye. Join LP, PE. Then, the angle under which the object is seen through the glass, is equal to PEQ; and the angle under which it is seen with the naked eye, is QLP; also, when these angles are small, since they have a common subtense QP, they are nearly in the inverse ratio of the radii EQ, LQ; that is, the visual angle when the object is seen through the glass : the visual angle when it is seen with the naked eye at the distance LQ :: $LQ : EQ$.

Ex. If the focal length of the glass be $\frac{1}{50}$ of an inch, and the least distance of distinct vision, eight inches, the visual angle of the object when viewed through the glass : the visual angle when it is seen with the naked eye :: $8 : \frac{1}{50}$ :: $400 : 1$.

In this microscope, the object appears erect (Art. 281).

(361.) The solar microscope is a single convex lens, used in the same manner as in the magic lantern (Art. 275). The moveable tube is adjusted to a hole in the window shutter of a darkened chamber; and the object to be examined, is strongly illuminated, and placed a little farther from the lens than it's principal focus; an inverted image of the object is thus formed, at a considerable distance from the lens, and received upon a screen placed at the concourse of the refracted rays.

The angles which the image and object subtend at the center of the eye, when viewed at the least distance of distinct vision, are proportional to their linear magnitudes, that is, to their distances from the center of the glass*.

On the double Microscope.

(362.) The astronomical telescope, when adapted to near objects, becomes a double microscope.

QP is an object, placed a little farther from the lens MN than it's principal focus F; qp the image of QP,

formed on the other side of the lens, and at a considerable distance from L it's center. AEB is a convex eye glass, whose axis coincides with the axis of

* On the construction, and use of the solar microscope, the reader may consult Mr. Adam's work, entitled '*Essays on the Microscope.*'

the lens MN, and whose distance from L is equal to the sum of Lq, and it's own focal length qE; consequently, the image qp is in the principal focus of the eye glass, and it may therefore be seen distinctly by a spectator whose eyes are able to collect parallel rays.

(363.) Since the conjugate foci, Q and q, move in the same direction upon the indefinite line QLO (Art. 189), if the glasses be fixed in a tube, or attached to each other in any other way, by moving the object QP, the image qp may be brought into the principal focus of the eye glass; or into such a situation, that the rays may, after the latter refraction, have a proper degree of divergency, or convergency for the eye of the spectator (See Art. 280).

Prop. LXXVIII.

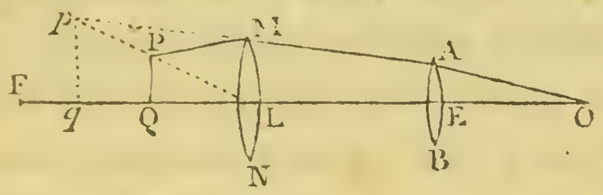
(364.) *To compare the angle which an object subtends at the center of the eye when seen through the double microscope, with the angle which it subtends at the naked eye when viewed at the least distance of distinct vision.*

Let qp (Fig. Art. 362) be the image of QP, formed in the principal focus of the lens AB, and terminated by the lines QLq, PLp. Then, the visual angle will be increased by the microscope on two accounts; first, because the image qp is greater than the object; and secondly, because this image is seen under a greater angle when viewed through the glass, than when viewed with the naked eye. Now, supposing QP and qp to be viewed with the naked eye, at the least distance of distinct vision, the visual angle of QP : the

visual angle of qp (:: QP : qp) :: QL : Lq. Also, the visual angle of qp, when thus viewed, : it's visual angle when seen through the glass AB :: qE : the least distance of distinct vision (Art. 360); therefore, by compounding these two proportions, the visual angle of QP, when viewed with the naked eye at the least distance of distinct vision, : the visual angle, when it is viewed with the microscope, :: $QL \times qE$: $Lq \times$ the least distance of distinct vision.

(365.) When the glasses are thus combined, the object appears inverted (Art. 307).

(366.) When a great magnifying power is not required, the object is placed between the glass MN



and it's principal focus; thus an erect image qp is formed, on the same side of the lens with the object; and if Eq be the focal length of the eye glass AB, the image may be seen distinctly.

The visual angle may be determined as in the preceding case.

One advantage of this construction is a greater field of view. The object also appears erect; and less confusion is produced by the spherical surfaces of the glasses, than would be caused by a single glass with the same magnifying power.

Prop. LXXIX.

(367.) *The density of rays in the bright part of the image of a given object, formed upon the retina by a refracting telescope, or double microscope, varies, nearly, as the aperture of the object glass directly, and the area of the picture upon the retina inversely.*

The density of rays in the image, varies directly as their number and inversely as the area over which they are diffused (Art. 7); that is, supposing the transmitting power to be given, and all the rays refracted at the object glass to be received by the eye, as the area of the aperture of the object glass directly, and the area of the image upon the retina inversely.

(368.) Cor. 1. The density also varies according to the same law, when reflectors are used; but the effects of reflectors and refractors are not here compared, because a much greater quantity of light is lost in reflection than in refraction.

(369.) Cor. 2. If F and f be the focal lengths of the object glass and eye glass, and A the linear aperture of the object glass, the density of rays in the picture upon the retina varies as $\frac{A^2 f^2}{F^2}$.

For, the visual angle of the object when seen with the naked eye : it's visual angle when seen through the telescope :: f : F; and since the object is given, the first term in the proportion is invariable; therefore the visual angle when the object is seen through the telescope, or the linear magnitude of the image upon the

retina, varies as $\frac{F}{f}$; and the area of the image, as $\frac{F^2}{f^2}$; therefore, the density of rays in that image, varies as $\frac{A^2 f^2}{F^2}$.

(370.) Cor. 3. In the same manner, if F be the focal length of the object metal in the Newtonian telescope, A it's linear aperture, and f the focal length of the eye glass, the density of rays in the picture upon the retina, if the reflecting power be given, varies as $\frac{A^2 f^2}{F^2}$.

(371.) The density of rays in the picture upon the retina is usually taken as a measure of the apparent brightness; though strictly speaking, apparent brightness has no numerical measure.

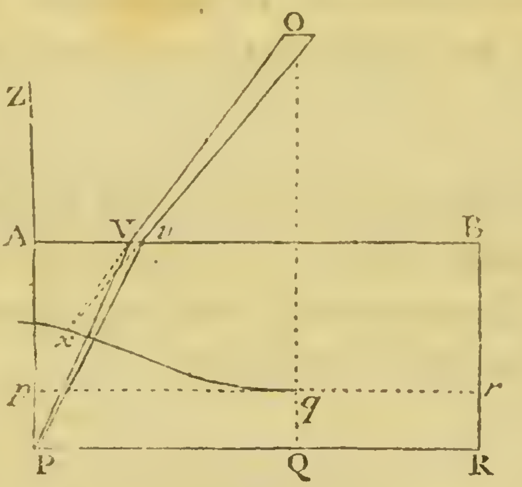
SCHOLIUM.

(372.) In explaining the construction and effects of optical instruments, we have supposed the images to be similar to the objects, and accurately formed in the geometrical foci of refracted or reflected rays. Were these suppositions true, telescopes and microscopes would be perfect; no limit could be set to their magnifying powers, but such as arise from the difficulty of forming spherical surfaces of proper dimensions; and by their assistance, objects might be seen as distinctly as if they were viewed in plane reflectors. The imperfections to which they are subject, arise from two causes; The spherical figure of the reflecting and refracting surfaces; and the unequal refrangibility of

the different rays which constitute the body of light by which objects are seen.

1. When any points in an object are seen by oblique pencils, that is, by such pencils as are not nearly perpendicular to the reflecting, or refracting surfaces, those points do not appear in the places determined by the constructions and calculations, hitherto given*.

This will be easily understood by considering the most simple case of refraction. Suppose AB to be a



plane refracting surface; PQR a straight line parallel to it; pqr the geometrical image of PQR, as determined Art. 203; O the place of the eye. Then, the rays which flow from P, after refraction at the surface AB, are diffused through all parts of the medium in which the eye is placed; and if those rays of the pencil, which pass through O, diverge accurately from p, they enter the eye as if they came from a real object there; and p is the *visible* image of P; but, when the eye is at any considerable distance from the perpendicular PAZ, the point P is seen by an oblique portion

* The case of objects seen by rays reflected at any plane surfaces is excepted

of the general pencil, as *PVO*. Those oblique rays do not diverge accurately, from *any* point; and therefore the visible image will be indistinct. But, if x be the place where, if produced backwards, they occupy the smallest space, we may suppose this to be the visible image of P; and the curve which is the locus of x, to be the visible image of the whole line PR*. Again, if a distorted image be formed by the object glass, or reflector of a telescope, different parts of it lie at different distances from the principal focus of the eye glass; and, if one part can be seen distinctly, the rest will appear confused. Instead of spherical surfaces, it has been proposed to adopt such as are generated by the revolution of the ellipse, parabola, or hyperbola; but, independent of the difficulty of grinding these surfaces, little advantage can be expected from them, as each surface, will only reflect, or refract, those rays accurately, which belong to one particular focus, and the aberrations, in other cases, will generally be greater than those produced by such surfaces as are of a spherical form.

2. Another, and more considerable cause of imperfection in optical instruments, is the unequal refrangibility of differently coloured rays. We have, in all our calculations, supposed light to be homogeneal;

* There is considerable difficulty in determining the *visible* image of an object, when seen by reflected, or refracted rays. In the preceding case, if we suppose the impression to be made by those rays which are incident in the plane perpendicular to the refracting surface, the equation to the curve which is the locus of x, rises to eight dimensions. If we suppose the impression to be made by those rays which are equally inclined to the refracting surface, the equation rises to four dimensions. See Newt. *Lect. Opt.* p. 1, Prop. viij.

that, whilst it passes out of one given medium into another, the sine of incidence bears the same invariable ratio to the sine of refraction. But, Sir Isaac Newton discovered that the common light by which objects are viewed, consists of rays which differ both in *colour* and *refrangibility*; and, that those rays which differ in colour, always differ in refrangibility; that is, if the sines of incidence be equal, the sines of refraction are different, though the mediums remain the same. Hence it follows, that if the image of an object be distinctly formed by the red, which are the least refrangible rays, at one particular distance from a refractor, a distinct image will be formed by rays which have a different degree of refrangibility, as the blue rays, at a different distance from it; thus, the rays of different colours, which flow from the same point, being collected at different distances from the refractors, a confused, and coloured image of that point, is necessarily produced upon the retina; or upon any screen which receives the refracted rays.

We are now to consider, in what manner these imperfections may, in some degree, be remedied. And we shall begin with the latter, which is of greater importance, as the errors it produces are much more considerable than those which arise from the spherical form of the surfaces; we may add moreover, that it's theory is more easily explained, and it's effects more likely to be corrected in practice.

Sect. VIII.

ON THE ABERRATIONS PRODUCED BY THE UNEQUAL REFRANGIBILITY OF THE RAYS OF LIGHT; AND BY THE SPHERICAL FORM OF REFLECTING AND REFRACTING SURFACES.

Prop. LXXX.

Art. (373.) *THE sun's light consists of rays which differ in refrangibility and colour.*

This important discovery was made by Sir Isaac Newton, who describes the experiment by which it is established, in the following words*.

" In a very dark chamber, at a round hole, about one third part of an inch broad, made in the shut of a window, I placed a glass prism, whereby the beam of the sun's light which came in at the hole, might be refracted upwards toward the opposite wall of the chamber, and there form a coloured image of the sun. The axis of the prism was perpendicular to the incident

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* *Optics,* B. I. P. I. Prop. ii.

rays. About this axis I turned the prism slowly, and saw the refracted light on the wall, or coloured image of the sun, first to descend, and then to ascend. Between the descent and ascent, when the image seemed stationary, I stopped the prism, and fixed it in that posture, that it should be moved no more. For in that posture, the refractions of the light at the two sides of the refracting angle, that is, at the entrance of the rays into the prism, and at their going out of it, were equal to one another*. The prism therefore being placed in this posture, I let the refracted light fall perpendicularly upon a sheet of white paper at the opposite wall of the chamber, and observed the figure and dimensions of the solar image formed on the paper by that light. This image was oblong and not oval, but terminated with two rectilinear and parallel sides, and two semicircular ends. On it's sides it was bounded pretty distinctly, but on it's ends very confusedly and indistinctly, the light there decaying and vanishing by degrees. The breadth of this image answered to the sun's diameter, and was about two inches and the eighth part of an inch, including the penumbra. For the image was eighteen feet and an half distant from the prism, and at this distance, that breadth, if diminished by the diameter of the hole in the window-shut, that is by a quarter of an inch, subtended an angle at the prism of about half a degree, which is the sun's apparent diameter. But the length of the image was about ten inches and a quarter, and the length of the rectilinear sides about eight inches; and the refracting angle of the prism,

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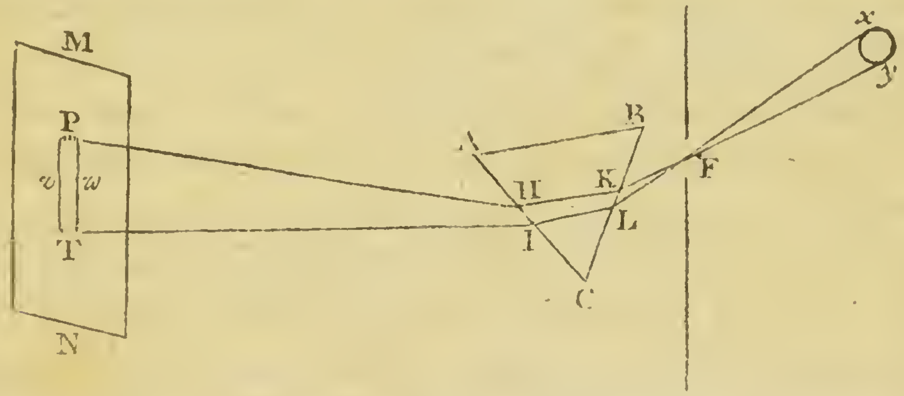
* *Lect. Opt.* P. I. Prop. xxv.

whereby so great a length was made, was 64 degrees. With a less angle the length of the image was less, the breadth remaining the same. It is farther to be observed, that the rays went on in right lines from the prism to the image; and therefore, at their very going out of the prism, had all that inclination to one another from which the length of the image proceeded, that is the inclination of more than two degrees and an half. And yet, according to the laws of optics vulgarly received, they could not possibly be so much inclined to one another.

" For let *F* represent the hole made in the window-shut, through which a beam of the sun's light was transmitted into the darkened chamber, and *ABC* a



triangular imaginary plane, whereby the prism is feigned to be cut transversely through the middle of the light; and let *xy* be the sun, *MN* the paper upon which the solar image or spectrum is cast, and *PT* the image itself; whose sides, towards *v* and *w*, are rectilinear and parallel, and ends, toward, *P* and *T*, semicircular. *yKHP*, and *xLIT* are two rays; the first of which comes from the lower part of the sun to the higher part of the image, and is refracted in the

prism at *K* and *H*; and the latter comes from the higher part of the sun to the lower part of the image, and is refracted at *L* and *I*. Since the refractions on both sides the prism are equal to one another, that is the refraction at *K* equal to the refraction at *I*, and the refraction at *L* equal to the refraction at *H*, so that the refractions of the incident rays at *K* and *L* taken together, are equal to the refractions of the emergent rays at *H* and *I* taken together; it follows, by adding equal things to equal things, that the refractions at *K* and *H* taken together, are equal to the refractions at *I* and *L* taken together; and therefore the two rays, being equally refracted, have the same inclination to one another after refraction, which they had before; that is, the inclination of half a degree, answering to the sun's diameter. So then, the length of the image *PT* would, by the rules of vulgar optics, subtend an angle of half a degree at the prism, and by consequence be equal to the breadth *vw*; and therefore the image would be round*. Since then it is found by experience that the image is not round, but about five times longer than broad, the rays which, going to the upper end *P* of the image, suffer the greatest refraction, must be more refrangible than those which go to the lower end *T*.

"The image or spectrum *PT* was coloured, being red at it's least refracted end *T*, and violet at it's most refracted end *P*, and yellow, green and blue in the intermediate spaces."

(374.) To shew that the unequal refrangibility of the rays in this experiment, is not accidental, or owing

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* See *Lect. Opt.* P. I. Sect. iv, &c.

to any new modification produced by the medium through which they pass, Sir Isaac Newton refracted the rays of each colour separately, and found that they ever after retained both their colour and peculiar degree of refrangibility*.

(375.) Nearly in the same manner, it may be shewn that common day light consists of rays which differ in colour and refrangibility.

For, if the round hole in the shutter receive only light from the clouds, and the eye be applied to the prism, the image is observed to be oblong, and coloured as in the former case.

(376.) Sir Isaac Newton, with the assistance of a person who had a more critical eye than himself, distinguished the spectrum into seven principal colours, proceeding from the less to the more refrangible rays, in the following order; red, orange, yellow, green, blue, indigo, violet; of which the yellow and orange were found to be the most luminous, and the next in strength were the red and green; the darker colours, especially the indigo and violet, affected the eye much less sensibly.

(377.) If, by any method, the prismatic colours be again united in the proportion which they have in the spectrum, they compound a white sun light; and by the mixture of different sorts of rays, in different proportions, various colours are produced, according to the quantity and nature of the rays united.

Thus, a mixture of red and yellow produces an orange; yellow and blue form a green†, &c.

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* Newt. *Optics,* Part I. Exp. 6.

† *Ibid,* P. I. Prop. iv.

(378.) From the former part of the last article we may conclude, that if a ray of white light be refracted through a medium contained by parallel planes whose distance is inconsiderable, it will not, as to sense, be separated into distinct colours.

For, the ray of each particular colour emerges parallel to the incident white ray; consequently, the emergent rays of different colours are parallel to each other; and since the thickness of the medium is inconsiderable, they emerge nearly at the same point; and therefore excite only the sensation of whiteness.

Thus it happens, that objects seen through common window glass do not appear coloured*.

(379.) The same may be said, if the emergent rays, after several refractions, be parallel to the incident white ray, and the points of emergence nearly coincide.

Prop. LXXXI.

(380.) *The more refrangible rays are more reflexible.*

A ray of light cannot, consistently with the general law of refraction, pass out of a denser medium into a rarer when the sine of incidence exceeds the limit determined by this proportion, sin. refraction : sin. incidence :: radius : sin. incidence, which is the limit sought (Art. 101); therefore, the greater the ratio of the sine of refraction to the sine of incidence, the less

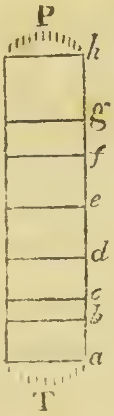
* Newt. *Lect. Opt.* P. II. Sect. iv.

will this limit be; and, consequently, the sooner will the rays be reflected*.

Prop. LXXXII.

(381.) *If a small cylindrical beam of white light pass nearly perpendicularly out of common glass into air, the dispersion of the differently coloured rays is about $\frac{2}{55}$ of the mean refraction.*

Let a small beam of the sun's light be refracted by a glass prism, in the manner described (Art. 373); and let PT represent the spectrum, divided by lines



which are perpendicular to it's parallel sides, and drawn through the confines of the several colours. Also, let ab, bc, cd, de, ef, fg, gh, be the spaces occupied by the red, orange, yellow, green, blue, indigo and violet rays, respectively; then if the whole length ah be represented by unity, ab is found to be $\frac{1}{8}$; $ac = \frac{1}{5}$; $ad = \frac{1}{3}$; $ae = \frac{1}{2}$; $af = \frac{2}{3}$; $ag = \frac{7}{9}$; and these are nearly

* This proposition may also be proved by experiment. Newt. *Optics*, B. I. P. I. Prop. iii.

proportional to the differences of the sines of refraction of the differently coloured rays, to a common sine of incidence.

Now, when the rays pass out of glass into air, if the common sine of incidence be represented by 50, the sines of refraction of the extreme red and violet rays are found to be 77 and 78 respectively* ; therefore the sines of refraction of the other rays, are $77\frac{1}{8}$, $77\frac{1}{5}$, $77\frac{1}{3}$, $77\frac{1}{2}$, $77\frac{2}{3}$, $77\frac{7}{9}$. That is, the sine of incidence of any red ray, is to the sine of refraction, in a ratio not greater than that of 50 : 77, nor less than that of 50 : $77\frac{1}{8}$; but varying, in different shades of red, through all the intermediate ratios. In the same manner, the sines of refraction of all the orange rays extend from $77\frac{1}{8}$ to $77\frac{1}{5}$, &c. the rays which are in the confines of the green and blue, have a mean degree of refrangibility, and the sine of incidence of these rays, is to the sine of refraction, as 50 to $77\frac{1}{2}$.

When the angles of incidence and refraction are small, they are nearly proportional to their sines ; and consequently, if the common angle of incidence be represented by 50, the deviation of the violet rays is 78 – 50, or 28 ; the deviation of the red rays is 77 – 50, or 27 ; therefore the difference of these, or the angle through which the rays of different colours are dispersed, is $\frac{1}{27}$ of the deviation of the red rays, $\frac{1}{28}$ of the deviation of the violet rays, and $\dfrac{1}{27\frac{1}{2}}$, or $\frac{2}{55}$ of the deviation of the rays of mean refrangibility, from their original course.

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* Newt. *Optics,* P. I. Prop. vii.

(382.) The angle through which all the red rays are dispersed is $\frac{1}{8}$ of $\frac{2}{55}$, or $\frac{1}{220}$ of the mean refraction, &c.

(383.) In general, if the sines of refraction of the red and violet rays, in their passage out of any given medium into air, be $1+m$ and $1+n$, to the common sine of incidence 1, then, when the angles of incidence and refraction are small, the dispersion of the rays is an $\frac{n-m}{m}$th part of the refraction of the red rays; and since m and n are invariable, this expression may be properly taken as the measure of the *dispersing* power of the medium.

(384.) Whilst the refracting mediums are the same, a given refraction of the mean rays is always attended with the same dispersion, which may be destroyed by an equal refraction in the opposite direction (Art. 29). But if the latter refraction fall short of the former, the dispersion will not be wholly corrected; if it exceed the former, the dispersion will be the contrary way; that is, the order of the colours will be changed; and no refraction can finally be produced by mediums of the same kind, without colour.

(385.) Mr. Dolland, an eminent optician in London, discovered*, about the year 1757, that different substances have different dispersing powers; that the same dispersion may be produced, or corrected, by a less refraction of the mean rays in one case, than in another; and thus refraction may, upon the whole, be produced without colour.

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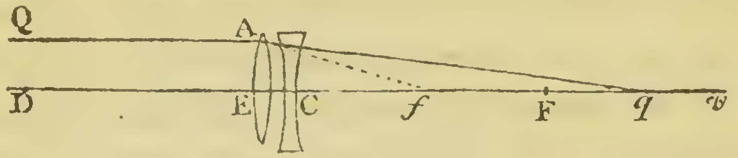
* The discovery has been ascribed to others; Mr. Dolland was the first who made it public.

Prop. LXXXIII.

(386.) *Having given the refracting powers of two mediums, to find the ratio of the focal lengths of a convex and concave lens, formed of these substances, which, when united, produce images nearly free from colour.*

Let $1+m : 1$, and $1+n : 1$ be the ratios of the sines of incidence and refraction of the red and violet rays out of air into the convex lens; $1+p : 1$, and $1+q : 1$, the ratios of those sines, out of air into the concave lens; F and f the focal lengths of the lenses, for red rays. Then $\frac{1}{m} : \frac{1}{n} :: F :$ the focal length of the convex lens for violet rays (Art. 173); therefore,



the focal length of the convex lens for violet rays is $\frac{mF}{n}$; in the same manner it appears, that $\frac{pf}{q}$ is the focal length of the concave lens for violet rays. Let AC be the compound lens; Eq it's focal length for red rays; Ev it's focal length for violet rays. Then $f-F:f :: F : Eq$; and $\frac{pf}{q}-\frac{mF}{n} : \frac{pf}{q} :: \frac{mF}{n} : Ev$ (Art. 192); hence $Eq=\frac{Ff}{f-F}$, and $Ev=\frac{mpFf}{npf-mqF}$; and when $Ev=Eq$, the red and violet rays, after both

refractions, are collected at q, or v. In this case, $\frac{Ff}{f-F}=\frac{mpFf}{npf-mqF}$; or $npf-mqF=mpf-mpF$; whence $p.\overline{n-m}.f=m.\overline{q-p}.F$; and $F:f::$ $p.\overline{n-m}:m.\overline{q-p}::\frac{n-m}{m}:\frac{q-p}{p}$. That is, the focal lengths are proportional to the dispersing powers of the two mediums. If the intermediate rays be dispersed according to the same law by the two mediums, it is manifest that the focal length of the compound lens, for these colours, will be Eq or Ev; and thus the image of a distant object will be formed in q or v, free from colour (Art. 377).

When the rays of different colours proceed from a point at a finite distance from this compound lens, after refraction they will converge to, or diverge from a common focus. For, the distance of the focus of refracted rays of any colour from the lens, depends upon the focal length of the lens, and the distance of the focus of incident rays from it (Art. 184); and since the latter quantities, by the supposition, are the same for rays of all colours, the distance of the focus of refracted rays from the lens, is the same; and thus, the image of an object at any finite distance from the compound lens, will be free from colour.

(387.) Ex. In crown, or common glass, $1+m=1.54$; and $1+n=1.56$. In flint glass, $1+p=1.565$; and $1+q=1.595$*; therefore the dis-

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* The refracting and dispersing powers of different kinds of glass are exceedingly various; and the causes upon which these powers depend are but imperfectly understood. See Dr. Blair's experiments on this subject, in the *Edinburgh Transactions*, vol. iii.

persing power of common glass : the dispersing power of flint glass :: $\frac{.02}{.54} : \frac{.03}{.565}$ :: $2 \times 565 : 3 \times 540$. To form a compound lens of these substances which shall produce a *real* image of a distant object, nearly free from colour, the convex lens must have the greater refracting power; and therefore it must be made of common glass, which has the less dispersing power. In this case, $F : f :: 2 \times 565 : 3 \times 540 :: 7 : 10$, nearly.

The focal length of the compound lens, $\frac{Ff}{f-F} = \frac{10F}{3}$.

(388.) Cor. 1. If the greater refraction be produced by the concave lens, it's focal length : the focal length of the convex lens :: 7 : 10, nearly; and the refracting power of the compound lens corresponds to that of a single concave glass.

(389.) Cor. 2. It is found by experience, that the extreme and intermediate rays are not dispersed by crown and flint glass, according to the same law; therefore, though the red and violet rays are united by the compound lens above described, yet the intermediate rays are not collected at the same point; and consequently, the images formed are not entirely free from colour.

The discovery of two sorts of glass, which shall disperse the extreme and intermediate rays in the same proportion, is still a desideratum in optics.

To form the most distinct image, the lenses ought to be so adjusted as to collect the brightest, and strongest colours, the yellow and orange.

(390.) Cor. 3. By a method similar to that employed in the proposition, two *compound* lenses, which collect the extreme rays, but disperse the intermediate rays in different proportions, might be so adjusted as to collect rays of three different colours, exactly; but the advantage thus gained, would probably not compensate for the loss of light.

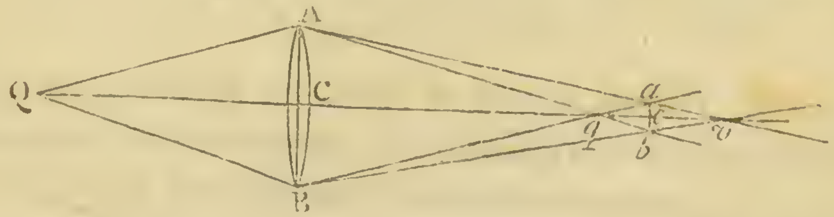
(391.) Cor. 4. Instead of a single convex lens, two are frequently employed, one on each side of the concave lens, which, when combined, have the same focal length with the single lens for which they are substituted. This construction lessens the aberration arising from the spherical form of the refracting surfaces.

Prop. LXXXIV.

(392.) *Having given the aperture of any lens, single or compound, and the foci to which rays of different colours, belonging to the same pencil, converge, to find the least circle of aberration through which these rays pass.*

Let QCv be the axis of the lens; AB, or $2\,AC$ it's linear aperture; q and v the foci of differently coloured rays. Draw Av, Bv; Aqb, Bqa; join a, b the points of their intersection, and let ab cut the axis QCv, in c.

Then, in the similar and equal triangles ACv, BCv, $Av=Bv$; and the $\angle\, AvC =$ the $\angle\, CvB$. In the



same manner, the $\angle\, AqC =$ the $\angle\, BqC$, or the $\angle\, bqc =$

the $\angle aqc$; therefore, in the triangles aqv, bqv, the angles avq, aqv are respectively equal to the angles bvq, bqv, and qv is common to both triangles, consequently aq is equal to bq. Hence it follows, that the triangles AqB, aqb, as also the triangles AqC, bqc, are similar, and that ab is perpendicular to Qcv; therefore ab is the diameter of the least circle of aberration, into which the rays converging to q and v are collected.

Now, from the similar triangles AqB, aqb, $AB : ab :: Aq : qb$; and from the similar triangles ACq, bqc, $Aq : bq :: Cq : cq$; therefore $AB : ab :: Cq : cq$. In the same manner, $AB : ab :: Cv : cv$; therefore $AB : ab :: Cv + Cq : cv + cq$ $(Cv - Cq)$.

(393.) Cor. 1. When the ratio of Cv to Cq is given, ab varies as AB; and the area of the least circle of aberration varies as AB^2.

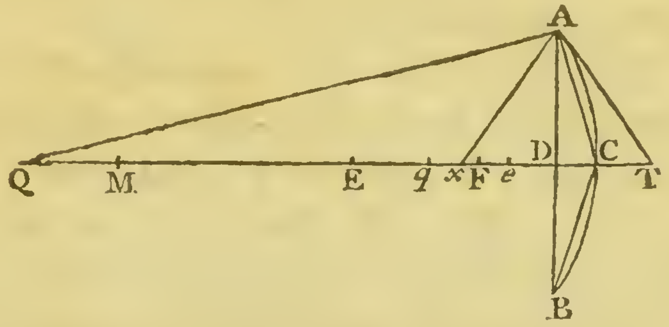
(394.) Cor. 2. Let parallel rays fall upon a single lens of crown glass, to compare the linear aperture of the lens, with the diameter of the least circle into which all the rays, of different colours, are collected.

Here $1 + m : 1 :: 1.54 : 1$; and $1 + n : 1 :: 1.56 : 1$ (Art. 387); and $Cv : Cq :: 56 : 54$ (Art. 173); therefore, $Cv + Cq : Cv - Cq :: 110 : 2 :: 55 : 1$; that is, $AB : ab :: 55 : 1$; or the diameter of the least circle of aberration into which the extreme rays, and consequently all the intermediate rays, are collected, is $\frac{1}{55}$ part of the linear aperture.

Prop. LXXXV.

(395.) *When a ray of light is incident obliquely upon a spherical reflector, to determine the intersection of the reflected ray and the axis of the pencil to which it belongs.*

Let ACB be the spherical reflector; E it's center; QA a ray incident obliquely upon it; QET the axis



of the pencil to which the ray belongs. Draw AT, touching the arc ACB in A, and let it meet QT in T; bisect ET, EC, in e and F; take q the geometrical focus, conjugate to Q; and let QA be reflected in the direction Ax; also, let Fe be small when compared with QF.

Then, $QF : FE :: QE : Eq$;

and $QF \pm Fe : FE + Fe :: QE : Ex$ (Art. 54);

therefore $Eq = \dfrac{QE \times FE}{QF}$; and $Ex = \dfrac{QE \times \overline{FE + Fe}}{QF \pm Fe}$,

whence, by actual division, $Ex = \dfrac{QE \times FE}{QF} + \dfrac{QE^2 \times Fe}{QF \times \overline{QF \pm Fe}} = \dfrac{QE \times FE}{QF} + \dfrac{QE^2}{QF^2} \times Fe$, nearly; therefore $Ex - Eq = qx = \dfrac{QE^2}{QF^2} \times Fe$, nearly.

Also, since $ET = 2\,Ee$, and $EC = 2\,EF$, CT, or $ET - EC = 2\,Fe$; consequently, $qx = \dfrac{QE^2}{QF^2} \times \dfrac{CT}{2}$ nearly.

(396.) This quantity, qx, is called the *longitudinal* aberration of the oblique ray QA.

(397.) COR. 1. Since the expression $\frac{QE^2}{QF^2} \times Fe$ has always the same sign, qx is always measured in the same direction upon the line QT.

(398.) COR. 2. Draw AD perpendicular to QC, and produce it till it meets the surface in B; join AC, CB. Then, the $\angle\, TAC$ = the $\angle\, CBA$ = the $\angle\, DAC$, and $CD : CT :: AD : AT$ (Euc. 3. vi.); and when the arc AC is evanescent, AD is equal to AT; therefore, $CD = CT$; and the longitudinal aberration $qx = \frac{QE^2}{QF^2} \times \frac{CD}{2}$.

(399.) COR. 3. When QA is parallel to QC, QE becomes equal to QF, and $qx = \frac{CD}{2}$.

(400.) COR. 4. If Q, when in FE, or in FE produced, approach to E, the ratio of QE to QF decreases; and therefore, if CD be given, the aberration decreases. If Q be in FC, or FC produced, as QF decreases, the aberration increases.

(401.) COR. 5. If the distances QE and QF be invariable, the aberration varies as CD.

(402.) COR. 6. Let CM be the diameter of the reflector; then, by the property of the circle, $CD : DA :: DA : DM$, and $CD = \frac{DA^2}{DM}$; therefore, when CD is very small, or DM nearly equal to the diameter of the given reflector, CD varies as DA^2, nearly; and consequently, when QE, QF are given, and the arc AC is very small, the longitudinal aberration varies as DA^2.

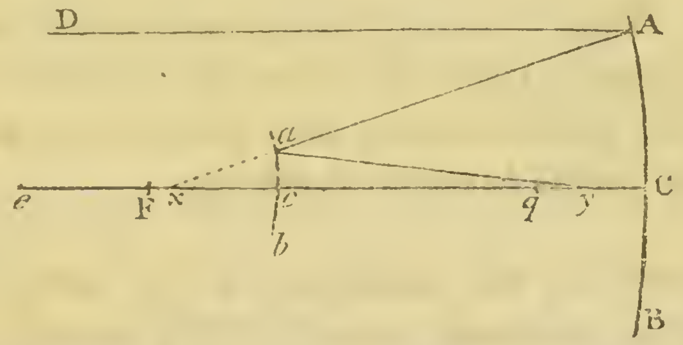
(403.) COR. 7. When parallel rays are incident upon the reflector, the longitudinal aberration is ultimately

equal to $\frac{CD}{2} = \frac{DA^2}{2\,DM} = \frac{DA^2}{4\,EC} = \frac{DA^2}{8\,EF}$; and therefore it varies as $\frac{DA^2}{EF}$.

Prop. LXXXVI.

(404.) *If parallel rays be reflected at a concave, and afterwards fall upon a convex spherical reflector, converging to a point between it's surface and principal focus, as in* Cassegrain's *telescope, the aberration of the lateral rays produced by the first reflection, will, in some measure, be corrected by the latter.*

Let ecC be the axis of the telescope; x the intersection of the axis and lateral ray after the first reflec-



tion; y their intersection after the second reflection; q the geometrical focus after both reflections.

The place of y, with respect to q, will be affected by two causes: 1st, x is nearer to c than F, the geometrical focus after the first reflection (Art. 397); and therefore, on this account, yc is less than qc (Art. 59); 2dly, in consequence of the aberration arising from the form of the reflector acb, yc is greater than

qc (Art. 397); therefore these causes counteract each other; and, by a proper adjustment of the reflectors, the aberration qy may, in a great measure, be destroyed.

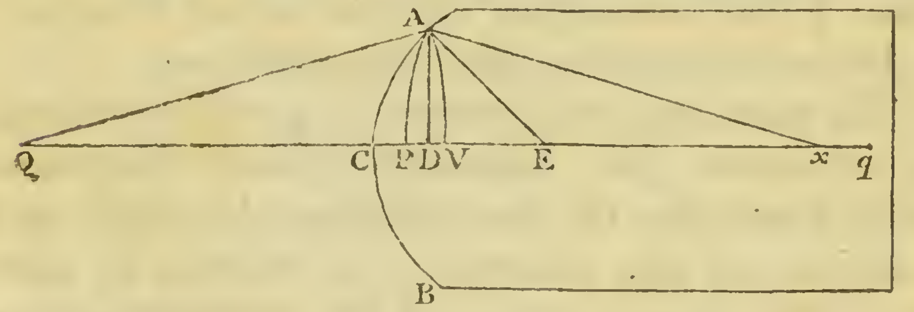
(405.) Cor. 1. Though the aberration qy, of the extreme ray DA, should be wholly destroyed, the aberration of the intermediate rays will not be entirely corrected; and this seems to be an insuperable obstacle to the perfection of reflecting telescopes.

(406.) Cor. 2. If the reflectors be both concave, as in Gregorie's telescope, the aberrations produced by the two reflections are in the same direction; that is, the second reflection increases the aberration produced by the first.

Prop. LXXXVII.

(407.) *When a ray of homogeneal light is incident obliquely upon a spherical refracting surface, to determine the intersection of the refracted ray and the axis of the pencil to which it belongs.*

Let ACB be the refractor; E it's center; QA a ray incident obliquely upon it; QCq the axis of the



pencil to which QA belongs; Ax the refracted ray;

q the geometrical focus, conjugate to Q. Draw AD perpendicular to the axis; and from the centers Q, x, with the radii QA, xA, describe the circular arcs AV, AP, cutting the axis in V and P. Take $m+1 : 1 ::$ sin. incidence : sin. refraction.

Then, in the triangle QAE, $QE : QA ::$ sin. incidence : sin. $\angle AEQ$; also, in the triangle AEx, $Ax : Ex ::$ sin. $\angle AEx$ (sin. $\angle AEQ$) : sin. refraction; therefore, by compounding these two proportions, $QE \times Ax : QA \times Ex ::$ sin. incidence : sin. refraction $:: m+1 : 1$; hence $\overline{m+1} \,.\, QA : QE :: Ax : Ex$; by division, $\overline{m+1} \,.\, QA - QE : QE :: Ax - Ex : Ex$*; that is, $\overline{m+1} \,.\, QV - QE : QE :: EP : Ex$; or $\overline{m+1} \,.\, QC + \overline{m+1} \,.\, CV - QE : QE :: EC - CP : Ex$; or $m \,.\, QC - EC + \overline{m+1} \,.\, CV : QE :: EC - CP : Ex$.

Now, $DC : DV :: QV : EC :: QC : EC$, nearly; and by composition, $DC : CV :: QC : QE$; therefore, when the arc AC is small, $CV = \dfrac{QE \times DC}{QC}$. Also $DC : DP :: Ax : EC$; by division, $DC : CP :: Ax : Ax - EC :: Ax : Ex :: \overline{m+1} \,.\, QC : QE$, nearly; therefore $CP = \dfrac{QE \times DC}{\overline{m+1} \,.\, QC}$, nearly. And, by substituting these values of CV and CP in the former proportion, we obtain $m.QC - EC + \dfrac{\overline{m+1} \,.\, QE \times DC}{QC}$:

* In this investigation of the aberration, diverging rays are supposed to fall upon a convex spherical surface of a denser medium, and to converge after refraction. The other cases may be derived from this, by a proper attention to the symbols.

$QE :: EC - \frac{QE \times DC}{\overline{m+1}\,.\,QC} : Ex$; hence, $Ex = QE \times$

$$\frac{EC - \frac{QE \times DC}{\overline{m+1}\,.\,QC}}{m.QC - EC + \frac{\overline{m+1}\,.\,QE \times DC}{QC}};$$ and by actually dividing, and taking the remainder, $Ex =$

$$\frac{QE \times EC}{m\,.\,QC - EC} - \frac{m\,.\,QE^2}{\overline{m+1}\,.\,QC} \times \frac{QC + \overline{m+2}\,.\,EC}{\overline{m\,.\,QC - EC}|^2} \times DC,$$

nearly. When DC vanishes, $Ex = Eq = \frac{QE \times EC}{m.QC - EC}$;

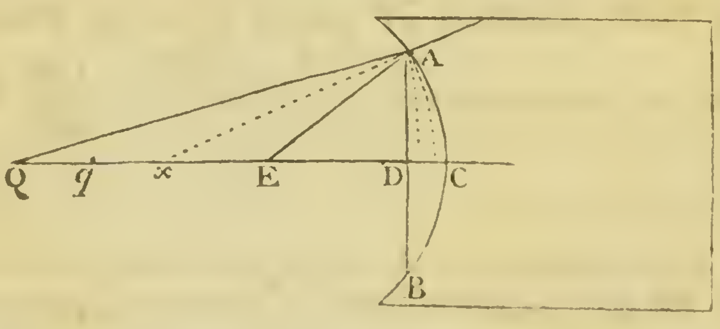
therefore the aberration $qx = \frac{m\,.\,QE^2}{\overline{m+1}\,.\,QC} \times$

$$\frac{QC + \overline{m+2}\,.\,EC}{\overline{m\,.\,QC - EC}|^2} \times DC.$$

(408.) Cor. 1. If the refractor be given, and the situation of the focus of incident rays, the aberration varies as DC, the versed sine of the arc AC.

(409.) Cor. 2. When the incident rays are parallel, QC becomes equal to QE; and $qx = \frac{DC}{m\,.\,\overline{m+1}}$.

(410.) Cor. 3. When diverging rays are incident upon a concave spherical refracting surface of a denser

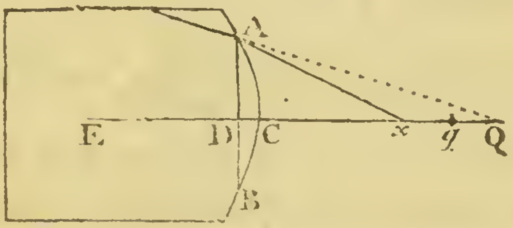


medium, the construction being made as before, EC

and DC are negative; whence, $Ex = -\frac{QE \times EC}{m.QC+EC} + \frac{m.QE^2}{\overline{m+1}.QC} \times \frac{QC - \overline{m+2}.EC}{\overline{m.QC+EC}|^2} \times DC$; and $qx = -\frac{m.QE^2}{\overline{m+1}.QC} \times \frac{QC - \overline{m+2}.EC}{\overline{m.QC+EC}|^2} \times DC$; this aberration, therefore, is to be measured in an opposite direction to the former.

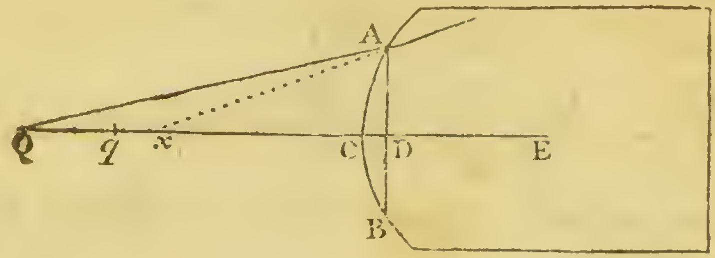
(411.) Cor. 4. In this last case, if $QC = \overline{m+2}.EC$, the aberration vanishes; that is, if $QC : EC :: m+2 : 1$, or $QE : EC :: m+1 : 1$ :: sin. incidence : sin. refraction (Art. 145).

(412.) Cor. 5. When converging rays are incident upon a concave spherical surface of a rarer medium,



QC, QE, EC and DC are negative. Also, if $1-\mu$: 1 :: sin. incidence : sin. refraction, $-\mu$ must be substituted for m, and $Ex = \frac{QE \times EC}{\mu.QC+EC} - \frac{\mu.QE^2}{\overline{1-\mu}.QC} \times \frac{QC + \overline{2-\mu}.EC}{\overline{\mu.QC+EC}|^2} \times DC$; hence, $qx = \frac{\mu.QE^2}{\overline{1-\mu}.QC} \times \frac{QC + \overline{2-\mu}.EC}{\overline{\mu.QC+EC}|^2} \times DC$.

(413.) Cor. 6. When diverging rays are incident upon a convex spherical surface of a rarer medium,



$$Ex = -\frac{QE \times EC}{\mu . QC + EC} + \frac{\mu . QE^2}{\overline{1-\mu} . QC} \times \frac{QC + \overline{2-\mu} . EC}{\overline{\mu . QC + EC}|^2} \times$$

DC; therefore the aberration $qx = -\frac{\mu . QE^2}{\overline{1-\mu} . QC} \times$

$$\frac{QC + \overline{2-\mu} . EC}{\overline{\mu . QC + EC}|^2} \times DC.$$

In the same manner, the aberration may be found in the other cases.

(414.) Cor. 7. Since $DC = \frac{DA^2}{2EC}$, nearly (Art. 402), by substituting this value of DC in the foregoing expressions, we obtain the aberration in terms of the semi-aperture.

(415.) Cor. 8. Since $Eq = \frac{QE \times EC}{m . QC - EC}$, if QE be diminished by the small quantity x, Eq will be increased by the quantity $\frac{\overline{m+1} . EC^2 \times x}{\overline{m . QC - EC}|^2}$. For on this supposition, Eq becomes $\frac{\overline{QE - x} . EC}{m . \overline{QC - x} - EC} =$

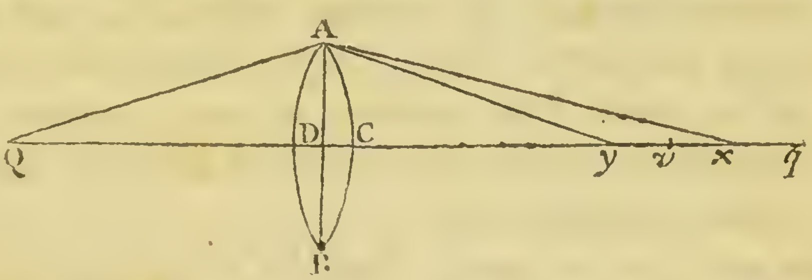
$$\frac{\overline{QE - x} . EC}{m . QC - EC - mx} = \frac{QE \times EC}{m . QC - EC} + \frac{\overline{m+1} . EC^2 \times x}{\overline{m . QC - EC}|^2}$$

nearly; and therefore $\frac{\overline{m+1}\,.\,EC^2 \times x}{m\,.\,\overline{QC-EC}^2}$, where x is the decrement of QE, is the increment of Eq, nearly.

(416.) Cor. 9. If x vary as DC, the increment of Eq, when the radius of the refractor and the situation of the focus of incident rays are given, will also vary as DC.

(417.) Cor. 10. By a proper application of the foregoing rules, the longitudinal aberration, arising from the spherical form of refracting surfaces, may be found in all cases where the apertures are small.

Ex. Let Qq be the axis of a lens; Q the focus of incident rays; q the geometrical focus after the first refraction, determined by Art. 137; v the geometrical



focus of emergent rays (Art. 194). Also, let QA be refracted, at the first surface, in the direction Ax, and emergent in the direction Ay. Then, the aberration vy arises from two causes; 1st, x does not coincide with the geometrical focus q (Art. 407); and since v is determined on supposition that q is the focus of rays incident upon the second surface, an aberration will be produced, which may be determined by Art. 415. 2dly, Ax is incident obliquely upon the latter surface, and the aberration arising from this cause may be

determined by Art. 412; therefore the whole aberration vy may be found.

(418.) Cor. 11. If the lens and place of the focus of incident rays be given, the aberration arising from each of these causes will vary nearly as AD^2 (Arts. 407. 416. 414); and therefore the final aberration vy, which is the sum or difference of the former, will also vary nearly as AD^2.

(419.) It is not consistent with the plan of this work to enter farther into these calculations; perhaps too much has been said already. The reader will find little difficulty in the application of the principles, if he wish to deduce practical rules for the construction of object glasses. Thus much it may be proper to observe, that the aberrations produced by a convex and concave lens are of contrary affections, and tend to correct each other; by a proper adjustment therefore, of the radii of the surfaces, a compound lens may be constructed, which will entirely destroy the aberration of the extreme rays*.

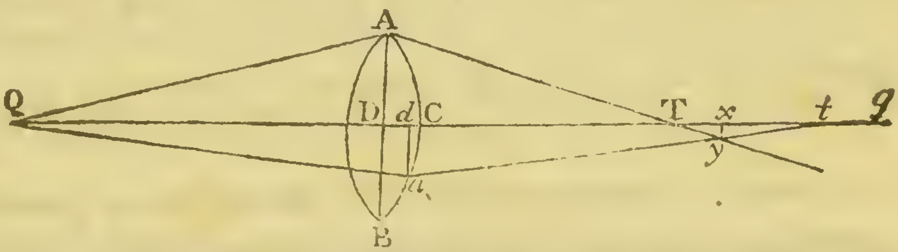
We may also observe, that the aberration is less, when two surfaces, or two lenses of the same kind are employed, than when the *same refraction* is produced by a single surface, or lens of the same description, and equal aperture.

* This subject is treated with great ability in the Encyclopædia Britannica, under the head Telescopes.

PROP. LXXXVIII.

(420.) *To find the least circle of aberration into which all the homogeneal rays of the same pencil, refracted by a lens or single surface, are collected.*

Let AB be the refractor; QCq it's axis; Q the focus of incident rays; T the intersection of the ex-



treme ray QAT, and the axis; t the intersection of any other ray Qat on the other side of QC, and the axis; y the intersection of ATy and at. Draw AD, ad, and yx at right angles to QCq; then, if the point a move from C towards B, the perpendicular xy will vary on two accounts: the increase of the angle Cta, and the decrease of the distance Tt; and when xy is a maximum, all the rays incident upon the same side of QC with Qa, will pass through it; and if the figure revolve about the axis Qq, all the rays incident upon the lens will pass through the circle generated by xy. It is also manifest, that the circle thus generated, is less than any other circle through which all the refracted rays pass. To find when xy is the greatest possible, let $Tx=x$; $ad=v$; $AD=a$; $DT=f$; $Tq=b$. Then, since AD, ad, when the lens is thin, are the semi-apertures through which the rays QAT, Qat pass, $AD^2 : ad^2 :: qT : qt$ (Art.

418); or, $a^2 : v^2 :: b : qt$; whence $qt = \frac{bv^2}{a^2}$; therefore $Tq - qt = Tt = b - \frac{bv^2}{a^2} = \frac{b}{a^2} \times \overline{a^2 - v^2}$. Again, $DT : DA :: Tx : xy$; or, $f : a :: x : xy$; consequently, $xy = \frac{ax}{f}$; also, $da : dt :: xy : tx$; or, $v : f :: \frac{ax}{f} : tx$; therefore $tx = \frac{ax}{v}$; hence, $Tx + xt = Tt = x + \frac{ax}{v} = \frac{b}{a^2} \times \overline{a^2 - v^2}$; or, $\frac{x}{v} \times \overline{a + v} = \frac{b}{a^2} \times \overline{a + v} \times \overline{a - v}$; and $x = \frac{b}{a^2} \times v \times \overline{a - v}$; consequently, x is the greatest possible, and therefore xy is the greatest possible, when $v \times \overline{a - v}$ is the greatest possible; or when $v = \frac{1}{2}a$. Hence it follows, that the greatest value of x is $\frac{b}{4}$; and the corresponding value of $xy = \frac{ab}{4f} = \frac{DA \times qT}{4\,DT}$.

(421.) Cor. 1. If the focal length of the refractor, and the focus of incidence, be given, DT is given, and $xy \propto qT \times DA \propto DA^3$ (Art. 418).

(422.) Cor. 2. On the same supposition, the area of the least circle of aberration varies as DA^6.

(423.) Cor. 3. Exactly in the same manner, we may find the least circle into which a pencil of rays, reflected by a spherical surface, is collected.

(424.) Cor. 4. When parallel rays are incident upon a spherical reflector, the longitudinal aberration varies directly as the square of the semi-aperture, and

inversely as the focal length (Art. 403); therefore, xy, the radius of the least circle of aberration, varies directly as the cube of the semi-aperture, and inversely as the square of the focal length of the reflector.

Prop. LXXXIX.

(425.) *The area of a circle of aberration in the image formed upon the retina by a telescope, or double microscope, varies directly as the area of the circle of aberration in the focus of the eye glass, and inversely as the square of the focal length of the eye glass.*

When the circle of aberration is in the principal focus of the glass through which it is viewed, it's visual angle is equal to the angle which it subtends at the center of the glass; and therefore, the *linear magnitude* of the circle of aberration upon the retina, varies as this angle (Art. 264); that is, it varies directly as the linear magnitude of the circle of aberration in the principal focus of the glass, and inversely as the focal length of the glass; consequently, the *area* of the circle of aberration on the retina, varies directly as the area of the circle of aberration in the focus of the eye glass, and inversely as the square of the focal length of the eye glass.

(426.) Cor. 1. In a reflecting telescope of Sir Isaac Newton's construction, if F be the focal length of the reflector, A it's semi-aperture, f the focal length of the eye glass, the radius of the circle of aberration in it's principal focus, varies as $\frac{A^3}{F^2}$ (Art. 424); and there-

fore the area of this circle varies as $\frac{A^6}{F^4}$; consequently, the area of the circle of aberration on the retina, varies as $\frac{A^6}{F^4 f^2}$.

(427.) Cor. 2. The area of the circle of aberration on the retina, has usually been considered as a measure of the apparent indistinctness of vision. And, though it is manifest that indistinctness admits of no numerical representation *, yet if the circle of aberration be the same in two cases, *cæteris paribus*, the indistinctness will be the same; and if the circle of aberration be greater in one case than in another, the indistinctness will also, *cæteris paribus*, be greater. For, the rays which proceed from one point in the object, are diffused over the circle of aberration, and consequently they are mixed with the rays which belong to as many different foci as there are sensible points in that circle; therefore, the greater the area of the circle, the greater must be the confusion, or indistinctness arising from this dispersion of the rays.

Prop. XC.

(428.) *To find on what supposition a given distant object appears equally bright, and equally distinct, when viewed with different reflecting telescopes of Sir Isaac Newton's construction.*

The notation in Art. 426 being retained; since

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* One degree of indistinctness can no more be said to be a multiple or part of another, than one degree of taste, or smell can be said to be the double, or half of another.

the brightness is given, $\frac{4A^2f^2}{F^2} \propto 1$ (Art. 370); or, $\frac{A^6f^6}{F^6} \propto 1$. Also, since the indistinctness is given $\frac{A^6}{F^4f^2} \propto 1$; therefore $\frac{A^6f^6}{F^6} \propto \frac{A^6}{F^4f^2}$; and $f^8 \propto F^2$; or, $f \propto F^{\frac{1}{4}}$.

Again, $\frac{A^4f^4}{F^4} \propto 1$; that is, $\frac{A^4F}{F^4} \propto 1$; or, $A^4 \propto F^3$; and $A \propto F^{\frac{3}{4}}$.

(429.) Cor. The magnifying power $\propto \frac{F}{f}$ (Art. 338); that is, as $F^{\frac{3}{4}}$.

SECT. IX.

ON THE RAINBOW.

PROP. XCI.

Art. (430.) *If two quantities bear an invariable ratio to each other, their corresponding increments are in the same ratio.*

Let X and Y be the two quantities; x and y their corresponding increments. Then, by the supposition, $X : Y :: X+x : Y+y$; and alternately, $X : X+x :: Y : Y+y$; by division, $X : x :: Y : y$; therefore, alternately, $X : Y :: x : y$.

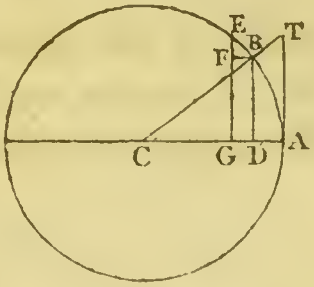
COR. Hence the increment of nx is n times the increment of x.

PROP. XCII.

(431.) *If the sines of two arcs be always in a given ratio, the evanescent increments of the arcs are proportional to the tangents of those arcs.*

Let AB be a circular arc whose radius is CA, sine BD, and tangent AT; draw EG parallel, and indefinitely near to BD, BF parallel to DG, and join EB.

Then, the triangle CBD is similar to the triangle EBF, formed by EF, FB, and the chord BE; for, the angles CDB, BFE are right angles; and the



$\angle EBC$ is a right angle (Newt. Princip. Lem. 6), and therefore equal to the $\angle FBD$; take away the common angle FBC, and the remaining angles, CBD and FBE are equal. Hence, $FE : BE :: CD : CB$; and in the similar triangles CDB, CAT, $CD : CA$ $(CB) :: DB : AT$; therefore, $FE : BE :: DB : AT$; whence $BE = \frac{FE \times AT}{DB}$; and BE is ultimately equal to the increment of the arc AB (Newt. Princip. Lem. 7); consequently, BE, the increment of the arc, $= \frac{FE \times AT}{DB}$; and since FE, the increment of the sine, varies as DB the sine (Art. 430), BE varies as AT.

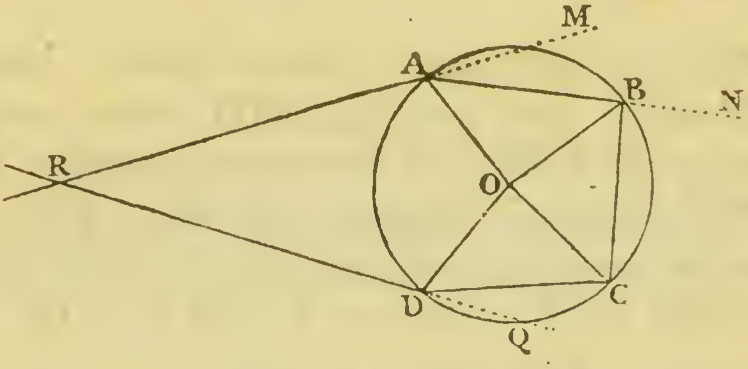
Prop. XCIII.

(432.) *If a ray of light refracted into a sphere, emerge from it after any given number of reflections, to determine the deviation of the ray, and the angle contained between the directions in which it is incident and emergent.*

Let a ray of light RA, incident upon the sphere $ABCD$ at A, be refracted in the direction AB; at B

let it be reflected in the direction BC; and at C, in the direction CD; at D let it be refracted out of the sphere, in the direction DR; produce RA, RD, to M, Q.

Take O the center of the sphere; join OA, OB, OC, OD; and let A be the angle of incidence of the ray RA; B the angle of refraction; R, a right angle.



Then the $\angle\, OAM = A$; the $\angle\, OAB =$ the $\angle\, OBA$ (Euc. 5. 1) $=$ the $\angle\, OBC$ (Art. 18) $=$ the $\angle\, OCB =$ the $\angle\, OCD =$ the $\angle\, ODC = B$. Also, the angles of deviation at A and D are equal; for if BA be supposed to be incident at A, the angle of incidence BAO, is equal to the angle of incidence CDO, of the ray CD; therefore the angles of deviation are equal*; and since the angle of deviation at A, is $A - B$, the whole deviation arising from the two refractions, is $2A - 2B$. Again, the angle of deviation at B is $2R - 2B$; and the angle of deviation, at every other reflection, is the same; therefore, if there be p reflections, the whole deviation, arising from this cause, is $2pR - 2pB$. To

* See Art. 25.

this, let the deviation arising from the refractions be added, and the whole deviation of the ray from it's original direction, is $2pR - 2\,.\,\overline{p+1}\,.\,B + 2A$.

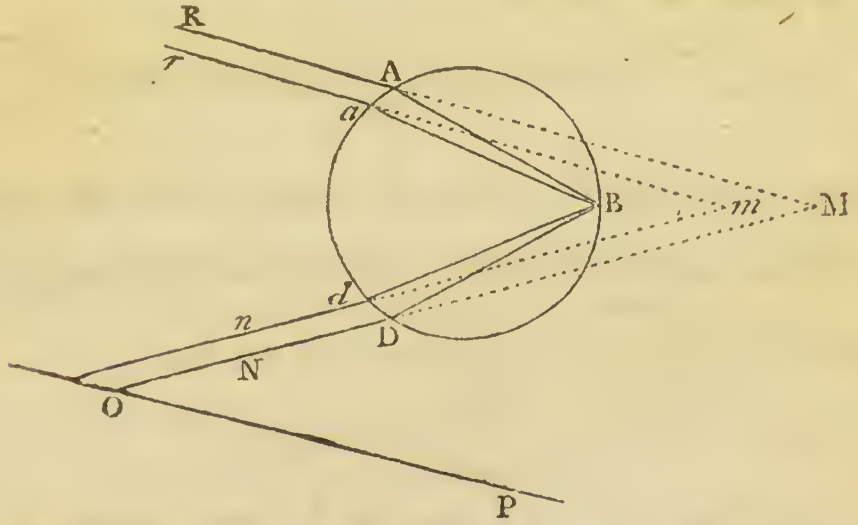
Also, a deviation through the angle $2pR$, which is a multiple of 180°, produces no inclination of the emergent to the incident ray; therefore, the inclination is represented by $2A - 2\,.\,\overline{p+1}\,.\,B$; or $2\,.\,\overline{p+1}\,.\,B - 2A$.

Prop. XCIV.

(433.) *If a small pencil of parallel homogeneal rays be refracted into a sphere, and the ratio of the sine of incidence to the sine of refraction be known, to find at what angle the rays must be incident, that they may emerge parallel after any given number of reflections within the sphere.*

Let RAM, ram, be the directions of the incident, DN, dn, the directions of the emergent rays; produce ND, nd, if necessary, till they meet RM, rm, in M and m.

Then, since AM, am, as also DN, dn, are parallel



by the supposition, the angles at M, and m are equal;

therefore, when the rays are incident at, or near to A, the angle RMN, contained between the incident and emergent ray, ceases to increase, or decrease; and therefore, the notation in the last article being retained, $2\,.\overline{p+1}\,.\,B-2\,A$, and consequently $\overline{p+1}\,.\,B-A$, ceases to increase, or decrease; that is, the increment of $\overline{p+1}\,.\,B$, is equal to the corresponding increment of A. Also, since sin. A is in a given ratio to sin. B, the increment of B : the increment of A :: tang. B : tang. A (Art. 431); or, multiplying the first and third terms by $p+1$, $\overline{p+1}\times$ increment of B : increment of A :: $\overline{p+1}$. tang. B : tang. A; and $\overline{p+1}\times$ increment of $B=$ increment of $\overline{p+1}\,.\,B$ (Art. 430); therefore, the increment of $\overline{p+1}\,.\,B$: the increment of A :: $\overline{p+1}$. tang. B : tang. A; and since the increment of $\overline{p+1}\,.\,B$ is equal to the increment of A, when the rays emerge parallel, $\overline{p+1}$. tang. $B=$ tang. A; or, tang. A : tang. B :: $p+1$: 1.

To determine the angles A and B, suppose x and y to be their cosines, the radius being unity; and let sin. A : sin. B :: m : n.

Then, $\sqrt{1-x^2}=$ sin. A; $\sqrt{1-y^2}=$ sin. B; $\frac{\sqrt{1-x^2}}{x}$ $=$ tang. A; $\frac{\sqrt{1-y^2}}{y}=$ tang. B. And, from the relation of the required angles, we have the following proportions; $\frac{\sqrt{1-x^2}}{x}:\frac{\sqrt{1-y^2}}{y}::p+1:1$;

and $\sqrt{1-y^2}:\sqrt{1-x^2}::n:m$;

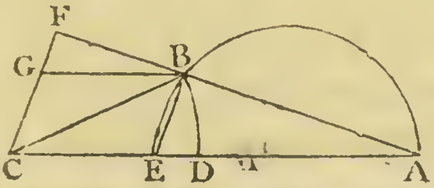
by composition, $\frac{1}{x}:\frac{1}{y}::n\,.\,\overline{p+1}:m$; hence, $y=$

$\frac{\overline{p+1}\,.\,nx}{m}$; and $y^2 = \frac{\overline{p+1}|^2\,.\,n^2x^2}{m^2}$; therefore, $1-y^2 : 1-x^2 :: 1 - \frac{\overline{p+1}|^2\,.\,n^2x^2}{m^2} : 1-x^2 :: n^2 : m^2$; and by multiplying extremes and means, $n^2 - n^2x^2 = m^2 - \overline{p+1}|^2\,.\,n^2x^2$; hence, $\overline{\overline{p+1}|^2 - 1}\,.\,n^2x^2 = m^2 - n^2$; or $\sqrt{p^2+2p}\,.\,nx = \sqrt{m^2-n^2}$; consequently, $1 : x :: \sqrt{p^2+2p}\,.\,n : \sqrt{m^2-n^2}$. The cosine of A being determined by this proportion, the angle itself may be found from the tables.

Also, $m : n ::$ sin. A : sin. B; and the three first terms in the proportion being known, the fourth is known; that is, sin. B is known; and therefore the angle B may also be found from the tables.

The angles A and B may also be determined by the following construction:

In the straight line $CEDA$, take CA to CD as m to n, and CA to CE as $p+1$ to 1; with the center C



and radius CD, describe an arc DB, cutting the circle ABE whose diameter is AE, in B; draw ABF; and join BC; then, the sine of the $\angle\, CBF$ will be to the sine of the $\angle\, CAF$ as m to n; and the tangent of CBF to the tangent of CAF as $p+1$ to 1; and consequently CBF, CAF will be the angles required.

Join BE, and complete the parallelogram $CEBG$, produce CG till it meets ABF in F. Then, in the

triangle CAB, sin. CBA (sin. CBF) : sin. CAB :: $CA : CB :: CA : CD :: m : n$. Again, since CF is parallel to EB, the $\angle\, BFG$, is equal to the $\angle\, EBA$, and is therefore a right angle; consequently, the lines FC, FG are tangents of the angles CBF, GBF (CAF) to the radius BF; and, in the similar triangles FCA, FGB, $FC : FG :: CA : GB :: CA : CE :: p+1 : 1$.

(434.) Ex. 1. If a small pencil of parallel red rays be incident upon a sphere of water, at an angle of about 59°. 23′, and suffer two refractions and one reflection, the rays will emerge parallel.

Here, $p=1$; and $m : n :: 108 : 81 :: 4 : 3$; therefore, $1 : x :: \sqrt{27} : \sqrt{7}$; or $x = \sqrt{\frac{7}{27}}$; and the angle whose cosine, to the radius unity, is $\sqrt{\frac{7}{27}}$, is 59°. 23′, nearly.

The angle of refraction B, whose sine is to the sine of 59°. 23′ :: 3 : 4, is 40°. 12′. Hence, the whole deviation, $2R-4B+2A$ (Art. 432), is 137°. 58′; which subtracted from 180°, gives the inclination of the incident, to the emergent pencil, 42°. 2′.

When violet rays are thus incident and emergent, $m : n :: 109 : 81$, and in this case, $A = 58°.\ 40'$; $B = 39°.\ 24'$; hence, $2R-4B+2A$ is 139°. 44′, and the inclination of the emergent, to the incident pencil, 40°. 16′.

(435.) Ex. 2. If parallel red rays fall upon a sphere of water, they will emerge parallel, after two refractions and two intermediate reflections, when the angle of incidence is about 71°. 50′.

In this case, $p=2$; and $1 : x :: \sqrt{72} : \sqrt{7}$; therefore the cosine of the angle of incidence is $\sqrt{\frac{7}{72}}$, which corresponds to an angle of $71°.50'$, nearly.

Also, $B=45°.27'$; and the whole deviation, $4R-6B+2A,=230°.58'$; hence, the inclination of the emergent, to the incident pencil, which is the excess of the whole deviation above $180°,=50°.58'$, nearly.

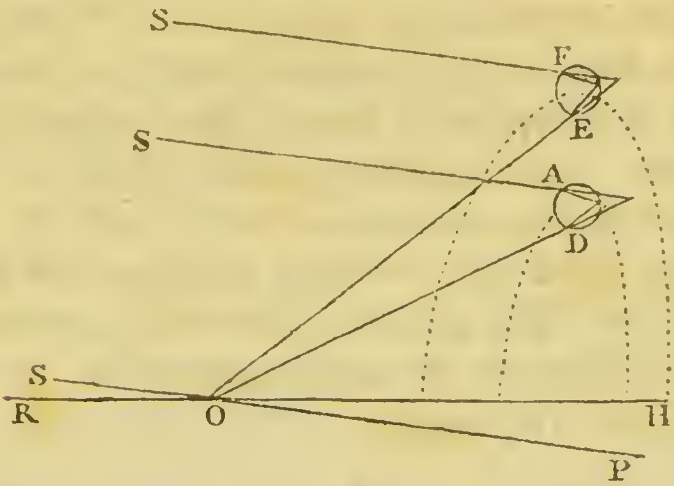
When violet rays are thus incident and emergent, $A=71°.26'$; $B=44°.47'$; $4R-6B+2A=234°.10'$; and the inclination of the emergent, to the incident pencil $=54°.10'$, nearly.

On the Formation of the Rainbow.

(436.) It has long been known that the rainbow is owing to the refraction and reflection of the sun's light by drops of rain. Antonius de Dominis first discovered that the interior, or primary bow, is caused by two refractions of the rays of light at each drop of water, and one reflection between them; and the exterior, or secondary bow, by two refractions and two reflections between them. This discovery he confirmed by experiments, which have been successfully repeated by more modern writers. If glass globes, filled with water, be placed in the sun's light, they may be elevated or depressed till they successively transmit to the eye, the colours of each bow, in their proper order*.

* Newton's *Optics*, Book I. Prop. ix.

(437.) To understand how the interior bow is formed, let O be the eye of a spectator; SOP a line passing through the eye and the sun. At the point



O, in the line PO, make the angle $POE = 42°.\ 2'$; then, when a drop of rain, FE, is in such a situation that the angle which OE makes with a perpendicular to it's surface at E is $59°.\ 23'$, a small pencil of parallel red rays will emerge from it at E, and enter the eye in the direction EO. For, if OE be considered as the incident pencil, it will emerge, after two refractions and one reflection, in the direction FS, which makes an angle of $137°.\ 58'$ with OE produced (Art. 434), or, an angle of $42°.\ 2'$ with OE, and is therefore parallel to OS; thus FS will pass through the sun*. Conversely, out of the beam of light proceeding from S, which falls upon the drop, the red rays incident at, and near to F, will, after two refractions and one reflection, emerge parallel, and entering the eye in the direction EO (Art. 29), will excite the sensation of their proper colour.

* The distance of the sun is so great, that two lines drawn from any points upon the surface of the earth, to a point in his disc, may be considered as parallel, in these calculations.

In the same manner, if OE revolve about the axis OP, every drop of water in the surface of the cone thus described, will transmit to the eye a small parallel pencil of red rays; and thus a red arc, whose radius, *measured by the angle which it subtends at the eye*, is $42^\circ.\ 2'$, will appear in the falling rain, opposite to the sun.

The other red rays of the beam which falls upon the drop FE, will, at their emergence, be inclined at different angles to the direction of the incident rays, and be so much dispersed before they reach the eye, and enter it in so weak a state, mixed with other rays, as to produce no distinct effect.

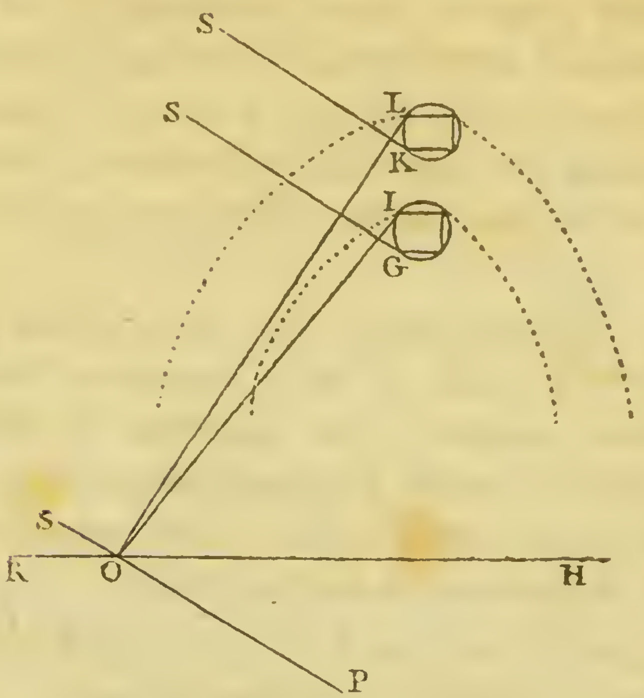
The parallel pencils of red rays, which emerge from other drops, fall above, or below the eye.

If the angle POD be $40^\circ.\ 16'$, and OD revolve about the axis OP, every drop of rain in the surface of the cone thus described, will transmit to the eye a parallel pencil of violet rays; and thus a violet arc will be formed, whose radius is $40^\circ.\ 16'$.

The drops between E and D will transmit to the eye parallel pencils of rays of different colours, orange, yellow, green, blue, indigo, in the order which they have in the prismatic spectrum (Art. 376); and the radii of the arcs of these respective colours may be calculated by the method employed in the 434th Article.

(438.) Again, let the angle $POI = 50^\circ.\ 58'$; and the angle $POL = 54^\circ.\ 10'$. Also, let OI, OL revolve about the axis OP. Then, it may be shewn as in the preceding case, that every drop of rain in the conical surface generated by OI, will transmit to the eye a

small parallel pencil of red rays, which has suffered



two refractions and two reflections, but sufficiently strong to excite the sensation of it's proper colour.

Also, every drop in the conical surface generated by *OL* will transmit to the eye a small pencil of parallel violet rays; and the intermediate drops, parallel pencils of rays of the intermediate colours. Thus the exterior bow is formed, in which the radii of the red and violet arcs are, respectively, 50°. 58′, and 54°. 10′.

The radii of the intermediate arcs may be determined by the method employed in the 435th Article.

(439.) Cor. 1. The colours in the two bows lie in a contrary order; the red forming the exterior ring of the primary, and the interior ring of the secondary bow.

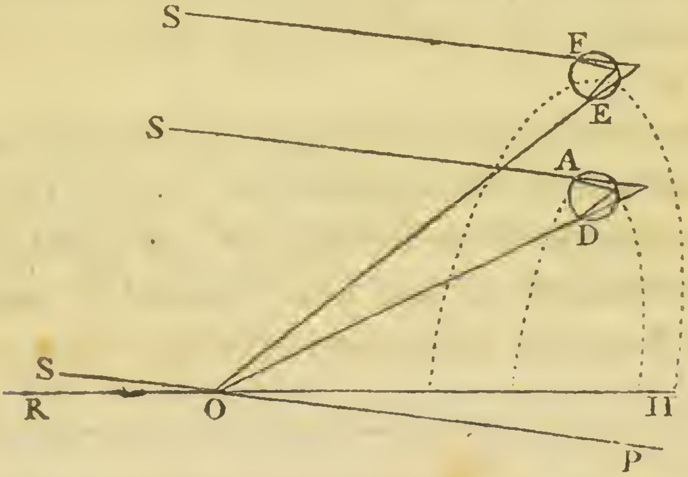
(440.) Cor. 2. Were the pencils sufficiently strong, a third bow, formed by two refractions and three reflections of the sun's rays in drops of rain, might be

seen. But, when the rays which are refracted into a drop of water, reach the farther surface, some of them pass out of the drop, and others are reflected within it. When these reflected rays again meet the surface, some of them pass out of the drop, and others suffer another reflection; and so on*. Thus the pencil becomes weaker at every reflection; and at length it contains so few rays as not to make a distinct impression upon the retina.

Prop. XCV.

(441.) *To find the altitude of the highest point of the rainbow, above the horizon, and the breadth of the colours.*

The construction being made as in the 437th Article, through O draw HOR parallel to the horizon.



Then the angle ROS, or HOP, measures the altitude of the point S above the horizon; and the altitude

* This is a fact, the cause of which has not been satisfactorily explained. Sir Isaac Newton supposes that rays of light, when they arrive at the surface of a medium, are sometimes in a state to be reflected, and sometimes to be transmitted; these states he calls fits

of the highest point of the red arc above the horizon, in the primary bow, is measured by EOH, or $EOP - HOP$, which is equal to $42°.\ 2' - HOP$. Also, the altitude of the highest point of the violet arc is measured by $DOP - HOP$, or $40°.\ 16' - HOP$. Hence it follows, that the breadth of the bow, supposing it to be formed by the rays which come from one point S, in the sun's disc, is $42°.\ 2' - 40°.\ 16'$, or, $1°.\ 46'$.

The breadth, thus determined, must be increased by $30'$, the sun's apparent diameter; for, the highest red arc is produced by the rays which flow from the lowest point in the sun's disc, and if ROS, or HOP, measure the altitude of the sun's center, the altitude of the highest red arc is $42°.\ 2' - HOP + 15'$; also, the lowest violet arc is produced by the rays which flow from the highest point in the sun's disc, and therefore the altitude of this arc, is $40°.\ 16' - HOP - 15'$; consequently, the breadth of the bow is $1°.\ 46' + 30'$, or $2°.\ 16'$.

In the same manner it appears, that the altitude of the violet arc, in the exterior bow, is $54°.\ 10' - SOR$; and the altitude of the red arc, $50°.\ 58' - SOR$; therefore, the breadth of the bow, formed by rays which proceed from any one point in the sun's disc, is $3°.\ 12'$.

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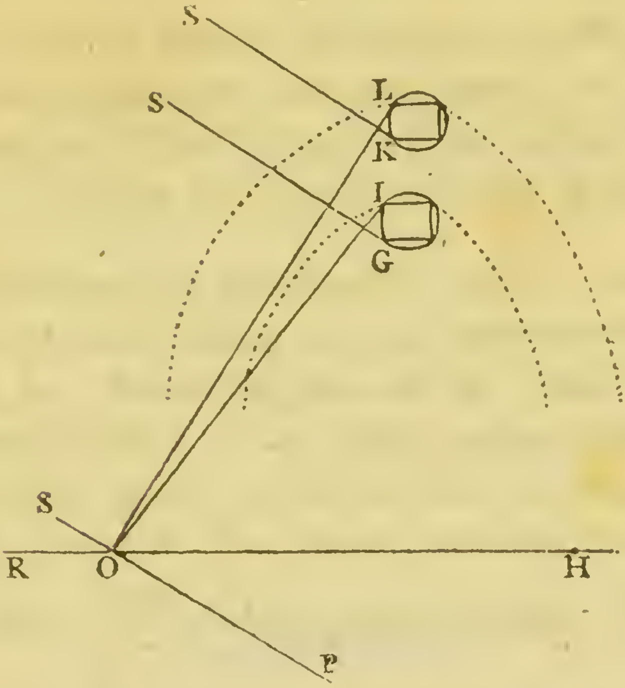
fits of easy reflection, and transmission; and accounts for them in the following manner: "Nothing more is requisite for putting the rays of light into fits of easy reflection, and easy transmission, than that they be small bodies, which by their attractive powers, or some other force, stir up vibrations in what they act upon; which vibrations being swifter than the rays, overtake them successively, and agitate them, so as by turns to increase and diminish their velocities, and thereby put them into those fits." *Opt.* Query 29.

If to this we add 30′, the sun's apparent diameter,



we have the actual breadth of the exterior bow = 3°. 42′.

(442.) Cor. 1. Since the altitude of an arc of any colour in the bow is equal to the radius of this arc diminished by the sun's altitude, when the sun is in the horizon, the altitude of the arc is equal to it's radius.

(443.) Cor. 2. The radius of any arc in the rainbow is equal to the altitude of the arc above the horizon, together with the sun's altitude.

(444.) Cor. 3. When the sun's altitude above the horizon, is equal to, or exceeds 42°. 2′, the primary bow cannot be seen; nor the secondary, when his altitude is equal to, or exceeds 54°. 10′.

Prop. XCVI.

(445.) *Having given the radius of an arc of any colour in the primary rainbow, to find the ratio of the sine of incidence to the sine of refraction when rays of that colour pass out of air into water.*

If A be the angle of incidence of the effective rays, B the angle of refraction, the radius of the arc is $4B - 2A$ (Art. 432); let the tangent of $2B - A$, half this angle, to the radius unity, be a; z the tangent of B. Then $2z = \text{tang.} A$ (Art. 433). Also, from the principles of trigonometry, tang. $2B : 2 \times \text{tang.} B\ (2z) :: 1^2 : 1^2 - z^2$; therefore tang. $2B = \frac{2z}{1-z^2}$. Again, tang. $\overline{2B - A}\ (a) : \text{tang.}\ 2B - \text{tang.} A \left(\frac{2z}{1-z^2} - 2z\right) :: 1^2 : 1^2 + \frac{4z^2}{1-z^2}$ *; hence, $\frac{2z}{1-z^2} - 2z = a + \frac{4az^2}{1-z^2}$; and by reduction, $2z^3 - 3az^2 - a = o$. The value of z being obtained † from this equation, the angles B and A, and consequently their sines, may be found from the tables.

(446.) Cor. In the same manner, if p be the number of reflections within the drop, z the tangent of B, Q

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* The propositions here referred to are the following;

1st, The tangent of the sum of two arcs, is to the sum of their tangents, as the square of radius, is to the square of radius diminished by the rectangle under the two tangents. 2d, The tangent of the difference of two arcs, is to the difference of their tangents, as the square of radius, to the square of radius increased by the rectangle under the two tangents. Mr. Vince's *Trig.* Art. 117.

† This equation has two impossible roots. See *Alg.* Art. 361.

the tangent of $\overline{p+1}\,.\,B$, a the tangent of $\overline{p+1}\,.\,B-A$, then $\overline{p+1}.z=$ tang. A; and $a : Q-\overline{p+1}\,.\,z :: 1^2 : 1^2 + \overline{p+1}\,.\,Qz$; therefore $Q-\overline{p+1}\,.\,z=a+\overline{p+1}\,.\,aQz$.

From which equation, the value of z being found, the angles A and B, and consequently their sines, may be determined by the tables.

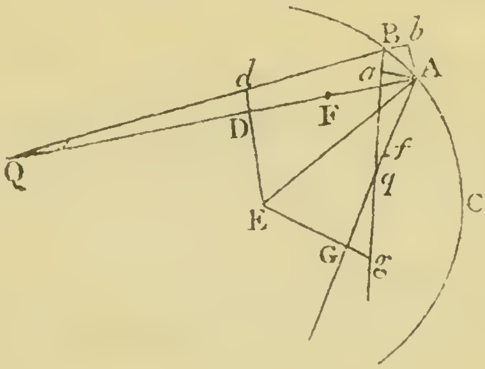
Sect. X.

ON CAUSTICS.

Prop. XCVII.

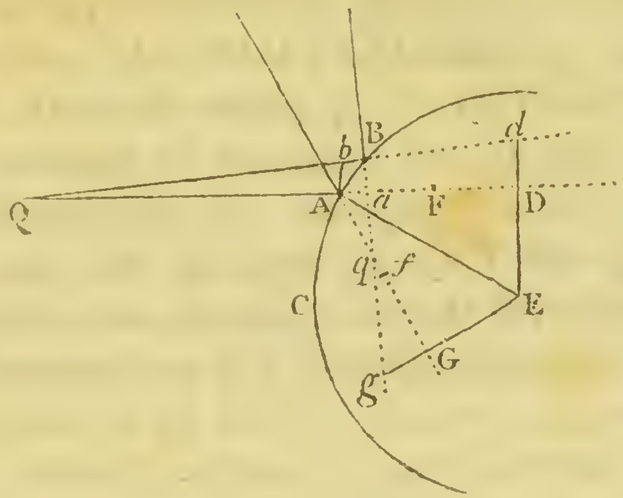
Art. (447.) *When a small pencil of diverging, or converging rays is incident obliquely upon a spherical reflector, in a plane which passes through it's center, to find the geometrical focus of reflected rays.*

Let BC be a spherical reflector whose center is E; QA, QB two rays of a small pencil incident obliquely



upon it, in the plane QAE; AG, Bg the reflected rays, or those rays produced backwards; q their intersection. From E, draw EDd, EGg at right angles to QA, AG; and when the arc AB is diminished without limit, they are also at right angles to QB, Bg; join EA, AB; from A, draw Ab, Aa at right angles to QB, qB, produced if necessary; bisect AD,

AG in F, f. Then, the angles EAD, EAG are the angles of incidence and reflection of the ray QA, or



equal to them; therefore they are equal to each other; the angles EDA, EGA are right angles; and the side EA is common to the two triangles EAD, EAG; consequently $AD=AG$; and $ED=EG$. In the same manner, $Ed=Eg$; whence, $Dd=Gg$. Again, in the evanescent triangles ABa, ABb, the angles ABb, ABa are complements to the angles of incidence and reflection of the ray QB, or equal to those complements; therefore they are equal to each other; also, the angles AbB, AaB are right angles; and AB is common to the two triangles; consequently, $Ab=Aa$.

Now, in the similar triangles QDd, QAb, $QA : QD :: Ab : Dd :: Aa : Gg$; and in the similar triangles qAa, qGg, $Aa : Gg :: Aq : qG$; therefore, $QA : QD :: Aq : qG$; whence, by composition, and division, $QA+QD : QA\sim QD :: Aq+qG : Aq\sim$

qG; that is, $2QF : 2FA :: 2Af : \overline{Af+fq} - \overline{Af-fq}$ $(2qf)$*; or, $QF : FA :: Af : fq$.

(448.) COR. 1. In the case represented by the first figure, QF and FA are measured in opposite directions from F; and since $QA : QD :: Aq : qG$, and QA is greater than QD, Aq is greater than qG; and therefore Af and fq are measured in opposite directions from f; hence it follows, that the equal rectangles $QF \times fq$ and $FA \times Af$ have, in this case, the same sign; therefore they will always have the same sign; that is, whenever QF and FA are measured in opposite directions from F, Af and fq are measured in opposite directions from f; and the contrary.

(449.) COR. 2. When the incident rays are parallel, FA is evanescent with respect to QF; therefore fq is evanescent with respect to Af; or, q coincides with f. Here, $Aq = \frac{1}{2} AG = \frac{1}{2} AD$.

(450.) COR. 3. If D be the focus of incident rays, G will be the focus of reflected rays. In this case, $QF = FA$; therefore $Af = fq$; and since QF and FA are measured in opposite directions from F, Af and fq must be measured in opposite directions from f; consequently, q coincides with G.

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* This conclusion depends upon the supposition that when QA and QD are measured in the same direction from Q, qA and qG are measured in opposite directions from q. If this be not the case, $Aq + qG = 2qf$; and $Aq \sim qG = 2fA = 2FA$; therefore $2qf = 2QF$, and $qf = QF$. Now let the rays be incident nearly perpendicularly upon the reflector, and F and f coincide with the principal focus; therefore Q and q are always equally distant from the principal focus, which is absurd (See Art. 50).

(451.) Cor. 4. If Q be a point in the circumference of the circle BC, $FA = \frac{1}{3}\,QF$; therefore $fq = \frac{1}{3}\,Af = \frac{1}{12}\,QA$; hence, $Aq = \frac{1}{4}\,QA + \frac{1}{12}\,QA = \frac{1}{3}\,QA$.

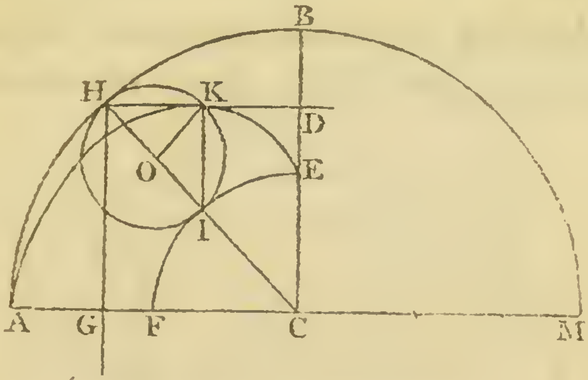
(452.) Cor. 5. The same propositions are true of any other reflecting curve, if E be the center of curvature of the evanescent arc AB.

(453.) Def. If an indefinite number of small pencils belonging to the focus Q, be incident, in the same manner, upon the reflecting surface BC, the curve which is the locus of the geometrical foci of reflected rays, is called the *caustic by reflection*.

Prop. XCVIII.

(454.) *To determine the form of the caustic, when the reflecting curve is a circular arc, and parallel rays are incident in the plane of the circle.*

Let C be the center of the proposed arc; CB, that radius of the circle which is parallel to the incident



rays; and ACM the diameter which is perpendicular to CB. Suppose GH, one of the incident rays, to be reflected in the direction HD; join CH; bisect CH

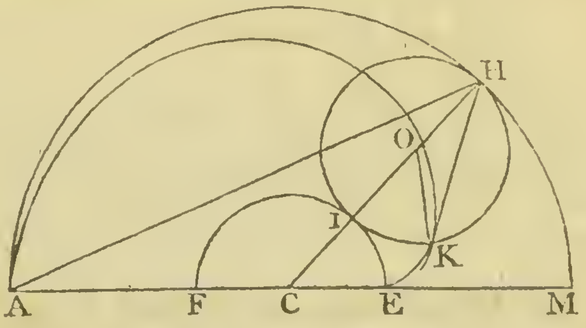
in I, and HI in O; with the centers C, O and radii CI, OH, describe the circles EIF, HKI; and let K be the intersection of HKI and HD; join OK, IK; and from C draw CD perpendicular to HD.

Then, since the angle HKI, in a semi-circle, is a right angle, the triangles HKI, HDC are similar; whence, $HD : HK :: HC : HI :: 2 : 1$; therefore K is a point in the caustic (Art. 449). Also, the $\angle KOI = 2 \angle IHK = 2 \angle CHG$ (Art. 18) $= 2 \angle ICE$ (Euc. 29. i.); and since circular arcs are as the angles which they subtend at their respective centers, and their radii jointly, the arc EI : the arc $IK :: 1 \times 2 : 2 \times 1$. Hence it follows, that the locus of the point K is an epicycloid, generated by the rotation of the circle HKI upon the circle EIF, in the plane of incidence AHM.

Prop. XCIX.

(455.) *To find the nature of the caustic, when the reflecting curve is a circular arc, and the focus of incident rays is in the circumference of the circle.*

Let AHM be the reflecting circle, C it's center, A the focus of incident rays; draw the diameter AM;



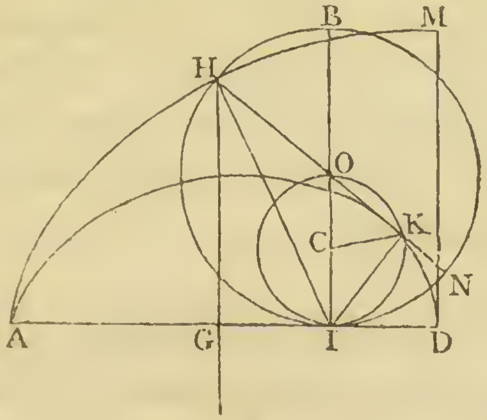
and let the ray AH be reflected in the direction HK; join CH, and divide it into three equal parts CI, IO,

OH; with the centers C and O, and radii CI, OI, describe the circles EIF, IKH; let K be the intersection of the reflected ray HK, and the circle HKI; join OK. Then, since the $\angle OKH$ = the $\angle OHK$ = the $\angle CHA$ = the $\angle CAH$, the triangles HCA, HOK are similar, and $HA : HC :: HK : HO$; alternately, $HA : HK :: HC : HO :: 3 : 1$; therefore K is a point in the caustic (Art. 451). Also, since the $\angle HOK$ = the $\angle HCA$, the $\angle ICE$ = the $\angle KOI$; and the radii CI, OI are equal; therefore the arcs EI, IK, are equal; and the locus of the point K is an epicycloid, generated by the rotation of the circle HKI upon the circle EIF, in the plane of the reflecting arc AHM.

Prop. C.

(456.) *To find the nature of the caustic, when the reflecting curve is a common cycloid, and the rays are incident parallel to it's axis.*

Let AHM be the reflecting semi-cycloid, whose base is AD, and axis DM; GH a ray of light incident



upon it at H; BHI the situation of the generating

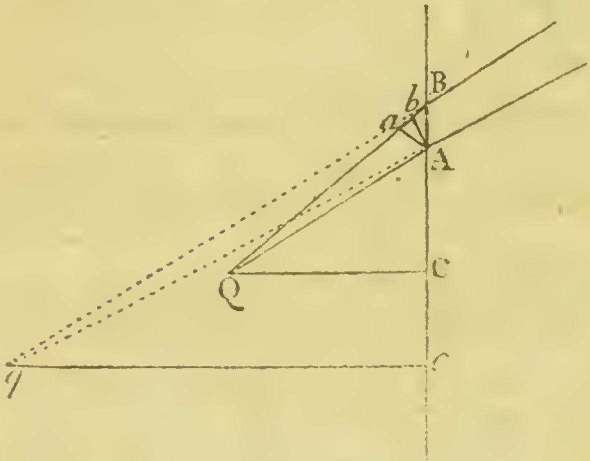
circle, when the point which traces out the cycloid is at H; and let I be the point in contact with the base. Take O the center of this circle, and draw the diameter HON; join OI; bisect the line OI in C, and with the center C and radius CI, describe the circle OKI, cutting HN in K; join IK, CK. Then, since OI is perpendicular to AD, or parallel to HG, the $\angle\, OIH =$ the $\angle\, GHI$; and the $\angle\, OIH =$ the $\angle\, OHI$; therefore, the $\angle\, OHI =$ the $\angle\, GHI$, and the ray GH is reflected in the direction HOK. Also, since IK is perpendicular to HK, and IH is half the radius of curvature of the cycloid at H (*Mech.* Art. 287), HK is one fourth part of the chord of curvature in the direction of the reflected ray; and therefore K is a point in the caustic (Art. 449). Again, since the $\angle\, KCI = 2\,\angle\, NOI$, and $OI = 2\,IC$, the arc $IK =$ the arc $IN = ID$; therefore, the locus of the point K is a common cycloid, whose base is AD, and generating circle OKI.

Prop. CI.

(457.) *When a small pencil of homogeneal rays falls obliquely upon a plane refracting surface, and in a plane which is perpendicular to that surface, having given the focus of incident rays, and the angles of incidence and refraction, to find the geometrical focus of refracted rays.*

Let BAC be the refracting surface; QA, QB, the extreme rays of the oblique pencil, incident in the plane of the paper; qA, qB produced, the directions in which they are refracted; q the intersection of the

refracted rays. From Q and q draw QC, qc at right angles to BC; and from A, draw, Aa, Ab, at right



angles to QB, qB. Take S and s to represent the sines of incidence and refraction of the ray QA; C and c their cosines; T and t their tangents. Then, since the angles AQC, BQC, are equal to the angles of incidence, and Aqc, Bqc, to the angles of refraction of the rays QA, QB, BQA and BqA are contemporaneous increments of the angles of incidence and refraction of the ray QA; and therefore, the $\angle BQA$: the $\angle BqA : T : t$* $:: \frac{S}{C} : \frac{s}{c}$. Also, the $\angle BQA$: the $\angle BqA :: \frac{Aa}{QA} : \frac{Ab}{qA}$; and since Aa, Ab are the cosines of the angles of incidence and refraction, to the radius BA, $\frac{C}{QA} : \frac{c}{qA}$ :: the $\angle BQA$: the $\angle BqA :: T : t :: \frac{S}{C} : \frac{s}{c}$; whence, $qA : QA :: \frac{T}{C} : \frac{t}{c} :: \frac{S}{C^2} : \frac{s}{c^2}$.

(458.) Cor. 1. Since $QA : QC :: r$ (radius) : C, we have $QA = \frac{r \times QC}{C}$. In the same manner, $qA =$

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* Art. 431.

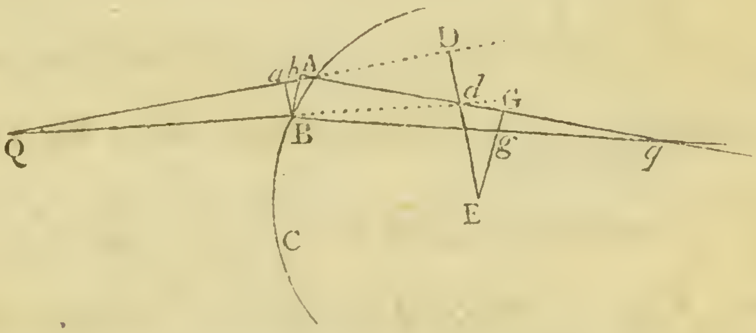
$\frac{r \times qc}{c}$; therefore $\frac{r \times qc}{c} : \frac{r \times QC}{C} :: \frac{T}{C} : \frac{t}{c}$; whence, $qc : QC :: \frac{T}{C^2} : \frac{t}{c^2}$.

(459.) Cor. 2. In the same manner it may be proved, that $QA = \frac{r \times AC}{S}$; and $qA = \frac{r \times Ac}{s}$; whence, $\frac{r \times Ac}{s} : \frac{r \times AC}{S} :: \frac{T}{C} : \frac{t}{c}$; consequently, $Ac : AC :: \frac{T}{S \times C} : \frac{t}{s \times c} :: \frac{1}{C^2} : \frac{1}{c^2} :: c^2 : C^2$.

Prop. CII.

(460.) *When a small pencil of homogeneal rays falls obliquely upon a spherical refractor, in a plane which passes through it's center, having given the focus of incident rays, and the angles of incidence and refraction, to determine the geometrical focus of refracted rays.*

Let QA, QB, be the extreme rays of a small pencil incident obliquely upon the spherical refractor ABC,



in a plane which passes through it's center E; and let Aq, Bq, be the refracted rays. Draw EdD, EgG, and Ba, Bb, at right angles to QA, qA, or to those

lines produced; and when the arc AB is diminished without limit, Ed and Eg, are at right angles to Qd, Bq. Take I and R to represent the angles of incidence and refraction of the ray QA. Then, since $ED : EG :: \text{sin. } I : \text{sin. } R :: Ed : Eg$, we have $Dd : Gg :: \text{sin. } I : \text{sin. } R$ (Euc. 19. v). Also, Ba, Bb are the cosines of I and R, to the radius AB.

From these two considerations, and the similarity of the triangles QDd, QaB; and qbB, qGg; we obtain the following proportions;

$$Dd : Gg :: \text{sin. } I : \text{sin. } R;$$
$$Gg : Bb :: Gq : bq\ (Aq);$$
$$Bb : Ba :: \text{cos. } R : \text{cos. } I;$$
$$Ba : Dd :: Qa\ (QA) : QD;$$

by compounding which proportions, we have $\text{sin. } I \times Gq \times \text{cos. } R \times QA = \text{sin. } R \times Aq \times \text{cos. } I \times QD$; and therefore, $Aq : Gq :: \frac{\text{sin. } I}{\text{cos. } I} \times QA : \frac{\text{sin. } R}{\text{cos. } R} \times QD ::$ $\text{tang. } I \times QA : \text{tang. } R \times QD$.

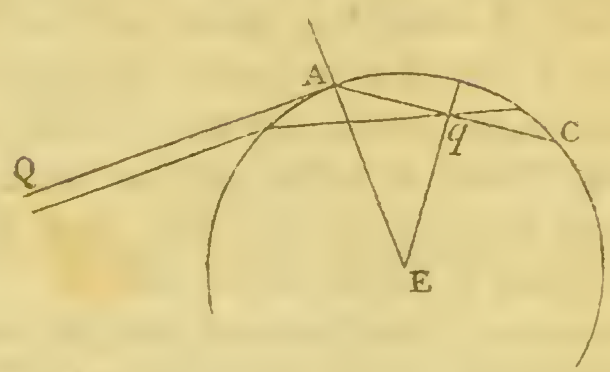
(461.) Cor. 1. The distances qA, qG, must be measured in the *same*, or *opposite* directions from q, according as QA, QD, are measured in the *same*, or *opposite* directions from Q*.

(462.) Cor. 2. When the incident rays are parallel, $QA = QD$, and therefore $Aq : Gq :: \text{tang. } I : \text{tang. } R$.

(463.) Cor. 3. On the foregoing supposition, when the rays pass out of a rarer medium into a denser, and the angle of incidence becomes nearly a right angle,

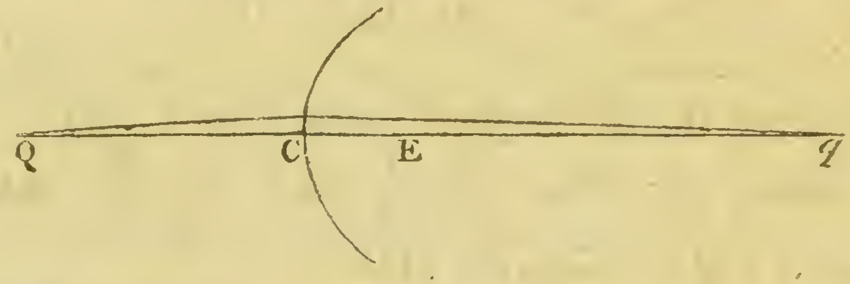
* See Art. 448, and Note, p. 242.

tang. I is indefinitely greater than tang. R; therefore



qG vanishes; or q bisects the chord of the arc, cut off by the refracted ray.

(464.) Cor. 4. When the rays are incident nearly perpendicularly upon the refracting surface, tang. I :



tang. R :: sin. I : sin. R; also, D and G coincide with E; therefore $qC : qE ::$ sin. $I \times QC$: sin. $R \times QE$.

(465.) Cor. 5. Similar conclusions may be drawn respecting the refraction of a small pencil of rays at any other surface, if E be the center of curvature of the refractor at the point of incidence.

On this subject, the reader may consult Hayes's *Fluxions*, Sect. ix. x. Smith's *Optics*, Book II. Chap. ix.

THE END.